

VARIATION AND OSCILLATION FOR SINGULAR INTEGRALS WITH ODD KERNEL ON LIPSCHITZ GRAPHS

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ABSTRACT. We prove that, for $\rho > 2$, the ρ -variation and oscillation for the smooth truncations of the Cauchy transform on Lipschitz graphs are bounded in L^p for $1 < p < \infty$. The analogous result holds for the n -dimensional Riesz transform on n -dimensional Lipschitz graphs, as well as for other singular integral operators with odd kernel. In particular, our results strengthen the classical theorem on the L^2 boundedness of the Cauchy transform on Lipschitz graphs by Coifman, McIntosh, and Meyer.

1. INTRODUCTION

The ρ -variation and oscillation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis (see [Lé], [Bo], [JKRW], [CJRW1], and [JSW], for example). The purpose of this paper is to establish some new results concerning the ρ -variation and oscillation for families of singular integral operators defined on Lipschitz graphs. In particular, our results include the L^p boundedness of the ρ -variation and the oscillation for the smooth truncations of the Cauchy transform and the n -dimensional Riesz transform on Lipschitz graphs, for $1 < p < \infty$ and $\rho > 2$.

Given a Borel measure μ in \mathbb{R}^d , one defines the n -dimensional Riesz transform of a function $f \in L^1(\mu)$ by $R^\mu f(x) = \lim_{\epsilon \searrow 0} R_\epsilon^\mu f(x)$ (whenever the limit exists), where

$$R_\epsilon^\mu f(x) = \int_{|x-y|>\epsilon} \frac{x-y}{|x-y|^{n+1}} f(y) d\mu(y), \quad x \in \mathbb{R}^d.$$

When $d = 2$ (i.e., μ is a Borel measure in \mathbb{C}), one defines the Cauchy transform of $f \in L^1(\mu)$ by $C^\mu f(x) = \lim_{\epsilon \searrow 0} C_\epsilon^\mu f(x)$ (whenever the limit exists), where

$$C_\epsilon^\mu f(x) = \int_{|x-y|>\epsilon} \frac{f(y)}{x-y} d\mu(y), \quad x \in \mathbb{C}.$$

To avoid the problem of existence of the preceding limits, it is useful to consider the maximal operators $R_*^\mu f(x) = \sup_{\epsilon>0} |R_\epsilon^\mu f(x)|$ and $C_*^\mu f(x) = \sup_{\epsilon>0} |C_\epsilon^\mu f(x)|$.

The Cauchy and Riesz transforms are two very important examples of singular integral operators with a Calderón-Zygmund kernel. The kernels $K : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$ that we consider in this paper satisfy

$$(1.1) \quad |K(x)| \leq \frac{C}{|x|^n}, \quad |\partial_{x^i} K(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{and} \quad |\partial_{x^i} \partial_{x^j} K(x)| \leq \frac{C}{|x|^{n+2}},$$

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for all $1 \leq i, j \leq d$ and $x = (x^1, \dots, x^d) \in \mathbb{R}^d \setminus \{0\}$, where $0 < n < d$ is some integer and $C > 0$ is some constant; and moreover $K(-x) = -K(x)$ for all $x \neq 0$ (i.e. K is odd). Notice that the n -dimensional Riesz transform corresponds to the vector kernel $(x^1, \dots, x^d)/|x|^{n+1}$, and the Cauchy transform to $(x^1, -x^2)/|x|^2$ (so, we may consider K to be any scalar component of these vector kernels).

Given an odd kernel K satisfying (1.1) and a finite Borel measure μ in \mathbb{R}^d , for each $\epsilon > 0$, we consider the ϵ -truncated operator

$$T_\epsilon \mu(x) = \int_{|x-y|>\epsilon} K(x-y) d\mu(y), \quad x \in \mathbb{R}^d,$$

and then we set $T\mu(x) = \lim_{\epsilon \searrow 0} T_\epsilon \mu(x)$ whenever the limit makes sense, and $T_*\mu(x) = \sup_{\epsilon>0} |T_\epsilon \mu(x)|$. Finally, given $f \in L^1(\mu)$, we define $T_\epsilon^\mu f(x) := T_\epsilon(f\mu)(x)$, $T^\mu f(x) := T(f\mu)(x)$ and $T_*^\mu f(x) := T_*(f\mu)(x)$. Thus, for a suitable choice of K , the operator T^μ coincides with the Cauchy or Riesz transforms.

Besides the operator T_ϵ defined above, one can consider another ϵ -truncated variant that we proceed to define. First we need some additional notation. Given $x = (x^1, \dots, x^d) \in \mathbb{R}^d$, we use the notation $\tilde{x} := (x^1, \dots, x^n) \in \mathbb{R}^n$. Let $\varphi_\mathbb{R} : [0, \infty) \rightarrow [0, \infty)$ be a non decreasing \mathcal{C}^2 function such that $\chi_{[3\sqrt{n}, \infty)} \leq \varphi_\mathbb{R} \leq \chi_{[2.1\sqrt{n}, \infty)}$ (the numbers $3\sqrt{n}$ and $2.1\sqrt{n}$ are chosen just for definiteness and they are not important). Given $\epsilon > 0$ and $x \in \mathbb{R}^d$, we denote

$$\varphi_\epsilon(x) := \varphi_\mathbb{R}(|\tilde{x}|/\epsilon) \quad \text{and} \quad \varphi := \{\varphi_\epsilon\}_{\epsilon>0}.$$

Given K as above, $x \in \mathbb{R}^d$, $0 < \epsilon$, and a finite Borel measure μ , we set

$$(K\varphi_\epsilon * \mu)(x) := \int \varphi_\epsilon(x-y) K(x-y) d\mu(y).$$

We also denote $(K\varphi * \mu)(x) := \{(K\varphi_\epsilon * \mu)(x)\}_{\epsilon>0}$. Finally, given $f \in L^1(\mu)$, we define $T_{\varphi_\epsilon}^\mu f(x) := (K\varphi_\epsilon * (f\mu))(x)$, $T_\varphi^\mu f(x) := \lim_{\epsilon \rightarrow 0} T_{\varphi_\epsilon}^\mu f(x)$ (whenever the limit makes sense), $T_{\varphi_*}^\mu f(x) := \sup_{\epsilon>0} |T_{\varphi_\epsilon}^\mu f(x)|$, and $\mathcal{T}_\varphi^\mu f(x) := \{T_{\varphi_\epsilon}^\mu f(x)\}_{\epsilon>0}$.

Let \mathcal{I} be a subset of \mathbb{R} (in this paper, we will always have $\mathcal{I} = (0, \infty)$ or $\mathcal{I} = \mathbb{Z}$), and let $\mathcal{F} := \{F_\epsilon\}_{\epsilon \in \mathcal{I}}$ be a family of functions defined on \mathbb{R}^d . Given $\rho > 0$, the ρ -variation of \mathcal{F} at $x \in \mathbb{R}^d$ is defined by

$$\mathcal{V}_\rho(\mathcal{F})(x) := \sup_{\{\epsilon_m\}} \left(\sum_{m \in \mathbb{Z}} |F_{\epsilon_{m+1}}(x) - F_{\epsilon_m}(x)|^\rho \right)^{1/\rho},$$

where the pointwise supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m \in \mathbb{Z}} \subset \mathcal{I}$. Fix a decreasing sequence $\{r_m\}_{m \in \mathbb{Z}} \subset \mathcal{I}$. The *oscillation* of \mathcal{F} at $x \in \mathbb{R}^d$ is defined by

$$\mathcal{O}(\mathcal{F})(x) := \sup_{\{\epsilon_m\}, \{\delta_m\}} \left(\sum_{m \in \mathbb{Z}} |F_{\epsilon_m}(x) - F_{\delta_m}(x)|^2 \right)^{1/2},$$

where the pointwise supremum is taken over all sequences $\{\epsilon_m\}_{m \in \mathbb{Z}} \subset \mathcal{I}$ and $\{\delta_m\}_{m \in \mathbb{Z}} \subset \mathcal{I}$ such that $r_{m+1} \leq \epsilon_m \leq \delta_m \leq r_m$ for all $m \in \mathbb{Z}$.

In this paper we are interested in studying the ρ -variation and oscillation for the family $\mathcal{T}_\varphi^\mu f$. That is, we will deal with

$$\begin{aligned} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu) f(x) &:= \mathcal{V}_\rho(\mathcal{T}_\varphi^\mu f)(x) = \mathcal{V}_\rho(K\varphi * (f\mu))(x) \quad \text{and} \\ (\mathcal{O} \circ \mathcal{T}_\varphi^\mu) f(x) &:= \mathcal{O}(\mathcal{T}_\varphi^\mu f)(x) = \mathcal{O}(K\varphi * (f\mu))(x), \end{aligned}$$

for a Borel measure μ and $f \in L^1(\mu)$. Although it is not clear from the definitions, these operators are μ -measurable (see [CJRW1], [JSW]).

Given $E \subset \mathbb{R}^d$, we denote by \mathcal{H}_E^n the n -dimensional Hausdorff measure restricted to E . Let $\Gamma := \{x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x}))\}$ be the graph of a Lipschitz function $A : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$ with Lipschitz constant $\text{Lip}(A)$. Let $H^1(\mathcal{H}_\Gamma^n)$ and $BMO(\mathcal{H}_\Gamma^n)$ be the (atomic) Hardy space and the space of functions with bounded mean oscillation, respectively, with respect to the measure \mathcal{H}_Γ^n . The following is our main result.

Theorem 1.1. *Let $\rho > 2$, let K be a kernel satisfying (1.1), and set $\mu := \mathcal{H}_\Gamma^n$. The operators $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ and $\mathcal{O} \circ \mathcal{T}_\varphi^\mu$ are bounded*

- in $L^p(\mu)$ for $1 < p < \infty$,
- from $H^1(\mu)$ to $L^1(\mu)$,
- from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, and
- from $L^\infty(\mu)$ to $BMO(\mu)$.

In all the cases above, the norm of $\mathcal{O} \circ \mathcal{T}_\varphi^\mu$ is bounded independently of the sequence that defines \mathcal{O} .

Let us recall that the $L^2(\mathcal{H}_\Gamma^1)$ boundedness of the Cauchy transform on Lipschitz graphs $\Gamma \subset \mathbb{C}$ with slope small enough was proved by A. P. Calderón in his celebrated paper [Ca]. The L^2 boundedness on Lipschitz graphs in full generality was proved later on by R. Coifman, A. McIntosh, and Y. Meyer [CMM].

Consider the Cauchy kernel $K(z) = 1/z$ ($z \in \mathbb{C}$), and set $\mu := \mathcal{H}_\Gamma^1$, so $C_\epsilon^\mu = T_\epsilon^\mu$. By standard Calderón-Zygmund theory (namely, Cotlar's inequality), the $L^2(\mu)$ boundedness of the Cauchy transform C^μ is equivalent to the $L^2(\mu)$ boundedness of the maximal operator C_*^μ . Let M^μ denote the Hardy-Littlewood maximal operator with respect to the measure μ . It is easy to check that, for $f \in L^1(\mu)$ with compact support, there exists some constant $C_0 > 0$ such that

$$C_\epsilon^\mu f(x) \leq T_{\varphi_\epsilon}^\mu f(x) + C_0 M^\mu f(x) \leq (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu) f(x) + C_0 M^\mu f(x)$$

for all $\epsilon > 0$, thus $(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu) + C_0 M^\mu$ controls the maximal operator C_*^μ and, in this sense, Theorem 1.1 (together with the known $L^p(\mu)$ boundedness of M^μ) strengthens the results of [Ca] and [CMM]. Analogous conclusions hold for the n -dimensional Riesz transform and the maximal operator R_*^μ .

The operator $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ is also related to an important open problem posed by G. David and S. Semmes which actually is our main motivation to prove Theorem 1.1. We need some definitions to state it.

Recall that a measure μ is said to be n -dimensional Ahlfors-David regular, or simply AD regular, if there exists some constant C such that $C^{-1}r^n \leq \mu(B(x,r)) \leq Cr^n$ for all $x \in \text{supp}\mu$ and $0 < r \leq \text{diam}(\text{supp}\mu)$. It is not difficult to see that such a measure μ must be of the form $\mu = h \mathcal{H}_{\text{supp}\mu}^n$, where h is some positive function bounded above and away from zero. A Borel set $E \subset \mathbb{R}^d$ is called AD regular if the measure \mathcal{H}_E^n is AD regular. One says that μ is uniformly n -rectifiable, or simply uniformly rectifiable, if there exist $\theta, M > 0$ so that, for each $x \in \text{supp}\mu$ and $R > 0$, there is a Lipschitz mapping g from the n -dimensional ball $B^n(0, R) \subset \mathbb{R}^n$ into \mathbb{R}^d such that $\text{Lip}(g) \leq M$ and $\mu(B(x, R) \cap g(B^n(0, R))) \geq \theta R^n$, where $\text{Lip}(g)$ stands for the Lipschitz constant of g . In the language of [DS2], this means that $\text{supp}\mu$ has big pieces of Lipschitz images of \mathbb{R}^n . A Borel set $E \subset \mathbb{R}^d$ is called uniformly n -rectifiable if \mathcal{H}_E^n is n -uniformly rectifiable. Of course, the n -dimensional graph of a Lipschitz function is uniformly n -rectifiable.

David and Semmes asked the following question, which is still open (see [Pa, Chapter 7]):

Problem 1.2. *Is it true that an n -dimensional AD regular measure μ is n -uniformly rectifiable if and only if R_*^μ is bounded in $L^2(\mu)$?*

It is proved in [DS1] that if μ is uniformly rectifiable, then R_*^μ is bounded in $L^2(\mu)$. However, the converse implication has been proved only in the case $n = 1$ and $d = 2$, by P. Mattila, M. Melnikov and J. Verdera [MMV], using the notion of curvature of measures (which seems to be useful only in this case).

Set $\mathcal{R}^\mu := \{R_\epsilon^\mu\}_{\epsilon>0}$. By combining some techniques from [DS2] and [To], in our forthcoming paper [MT] we show that the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ implies that μ is uniformly n -rectifiable. Moreover, we also prove that $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$ for all AD regular uniformly n -rectifiable measures μ . So we obtain the following theorem, which might be considered as a first approach to a possible solution of Problem 1.2:

Theorem 1.3. *Let $\rho > 2$. An n -dimensional AD regular measure μ is uniformly n -rectifiable if and only if $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is a bounded operator in $L^2(\mu)$.*

An essential ingredient for the proof of this result is Theorem 1.1 above. The arguments and techniques used to derive the L^2 boundedness of $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ on uniformly rectifiable measures from the L^2 boundedness of $\mathcal{V}_\rho \circ \mathcal{R}_\varphi^\mu$ on Lipschitz graphs are quite delicate (\mathcal{R}_φ^μ is defined as \mathcal{R}^μ but using the family φ for the truncations). In particular, they involve the corona type decomposition introduced in [DS1]. For this reason, the proof of the preceding theorem is out of the scope of this paper and will appear in [MT].

Concerning the background on the ρ -variation and oscillation, a fundamental result is Lépinglé's inequality [Lé], from which the L^p boundedness of the ρ -variation and oscillation for martingales follows, for $\rho > 2$ and $1 < p < \infty$ (see Theorem 2.4 below for more details). From this result on martingales, one deduces that the ρ -variation and oscillation are also bounded in L^p for the averaging operators (also called differentiation operators, see [JKRW]):

$$(1.2) \quad D_\epsilon f(x) = \frac{1}{|B(x, \epsilon)|} \int_{B(x, \epsilon)} f(y) dy, \quad x \in \mathbb{R}.$$

As far as we know, the first work dealing with the ρ -variation and oscillation for singular integral operators is the one of J. Campbell, R. L. Jones, K. Reinhold and M. Wierdl [CJRW1], where the L^p and weak L^1 boundedness of the ρ -variation (for $\rho > 2$) and oscillation for the Hilbert transform was proved. Recall that, for $f \in L^p(\mathbb{R})$ and $x \in \mathbb{R}$,

$$H_\epsilon f(x) = \frac{1}{\pi} \int_{|x-y|>\epsilon} \frac{1}{x-y} f(y) dy,$$

and then the Hilbert transform of f is defined by $Hf(x) = \lim_{\epsilon \rightarrow 0} H_\epsilon f(x)$, whenever the limit exists. Later on, there appeared other papers showing the L^p boundedness of the ρ -variation and oscillation for singular integrals in \mathbb{R}^d ([CJRW2]), with weights ([GT]), or for other operators such as the spherical averaging operator or singular integral operators on parabolas ([JSW]). Finally, we remark that, very recently, the case of the Carleson operator has been considered too ([LT], [OSTTW]).

Notice that the Hilbert transform is one of the simplest examples where Theorem 1.1 applies (one sets $\Gamma = \mathbb{R}$, i.e. $A \equiv 0$), and so one obtains a new proof of the L^p boundedness of the ρ -variation and oscillation for the Hilbert transform. In the original proof in [CJRW1], a key ingredient was the following classical identity, which follows via the Fourier transform:

$$(1.3) \quad Q_\epsilon = P_\epsilon * H,$$

where P_ϵ is the Poisson kernel and Q_ϵ is the conjugated Poisson kernel. Using this identity and the close relationship between the operators Q_ϵ and H_ϵ , Campbell *et al.* derived the L^p boundedness of the ρ -variation and oscillation for the Hilbert transform from the one of

the family $\{D_\epsilon(Hf)\}_{\epsilon>0}$, where D_ϵ is the averaging operator in (1.2) (notice that P_ϵ can be written as a convex combination of operators D_δ , $\delta > 0$).

In most of the previous results concerning ρ -variation and oscillation of families of operators from harmonic analysis, the Fourier transform is a fundamental tool. However, this is not useful in order to prove Theorem 1.1, since the graph Γ is not invariant under translations in general. Moreover, even for the Cauchy transform, there is no formula like (1.3), which relates the truncations of a singular integral operator with an averaging operator applied to a singular integral operator, when Γ is a general Lipschitz graph.

The main ingredients of our proof of Theorem 1.1 are the known results on the ρ -variation and oscillation for martingales (Lépingle's inequality [Lé]) and a multiscale analysis which stems from the geometric proof of the L^2 boundedness of the Cauchy transform on Lipschitz graphs by P. W. Jones [Jn1] and his celebrated work [Jn2] on quantitative rectifiability in the plane, using the so called β coefficients. Some of the techniques in these papers were further developed in higher dimensions by David and Semmes [DS1] for Ahlfors-David regular sets. More recently, in [To] some coefficients denoted by α , in the spirit of the Jones' β 's, were introduced, and they were shown to be useful for the study of the L^p -boundedness of Calderón-Zygmund operators on Lipschitz graphs and on uniformly rectifiable sets (see the definition below Theorem 1.4). In our paper, the α and β coefficients play a fundamental role.

Let us remark that Lépingle's inequality, which asserts the L^p boundedness of the ρ -variation of martingales, fails if one assumes $\rho \leq 2$ (see [Qi] and [JW], for example). Moreover, this fact can be brought to the ρ -variation of averaging operators and singular integral operators, thus it is essential to assume $\rho > 2$ in Theorem 1.1. Analogous conclusions hold if one replaces the ℓ^2 -norm by and ℓ^p -norm with $\rho < 2$ in the definition of \mathcal{O} . See [CJRW1], or [AJS] for the case of martingales.

Concerning the direct applications of Theorem 1.1, it is easily seen that the L^p boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^\mu$ yields a new proof of the existence of the principal values $T_\varphi^\mu f(x) := \lim_{\epsilon \rightarrow 0} T_{\varphi_\epsilon}^\mu f(x)$ for all $f \in L^p(\mu)$ and almost all $x \in \Gamma$, without using a dense class of functions in $L^p(\mu)$ as in the classical proof. Moreover, from Theorem 1.1 one also gets some information on the speed of convergence. In fact, a classical result derived from variational inequalities is the boundedness of the λ -jump operator $N_\lambda \circ \mathcal{T}_\varphi^\mu$ and the (a, b) -upcrossings operator $N_a^b \circ \mathcal{T}_\varphi^\mu$. Given $\lambda > 0$, $f \in L_{loc}^1(\mu)$ and $x \in \mathbb{R}^d$, one defines $(N_\lambda \circ \mathcal{T}_\varphi^\mu)f(x)$ as the supremum of all integers N for which there exists $0 < \epsilon_1 < \delta_1 \leq \epsilon_2 < \delta_2 \leq \dots \leq \epsilon_N < \delta_N$ so that

$$|T_{\varphi_{\epsilon_i}}^\mu f(x) - T_{\varphi_{\delta_i}}^\mu f(x)| > \lambda$$

for each $i = 1, \dots, N$. Similarly, given $a < b$, one defines $(N_a^b \circ \mathcal{T}_\varphi^\mu)f(x)$ to be the supremum of all integers N for which there exists $0 < \epsilon_1 < \delta_1 \leq \epsilon_2 < \delta_2 \leq \dots \leq \epsilon_N < \delta_N$ so that $T_{\varphi_{\epsilon_i}}^\mu f(x) < a$ and $T_{\varphi_{\delta_i}}^\mu f(x) > b$ for each $i = 1, \dots, N$. Using Theorem 1.1 one obtains (by the same arguments as in [CJRW1, Theorem 1.3 and Corollary 7.1]) the following:

Theorem 1.4. *Let $\rho > 2$, $\lambda > 0$, and let K , and μ be as in Theorem 1.1. For $1 < p < \infty$, there exist constants C_1 and C_2 depending on ρ , n , d , K , and $\text{Lip}(A)$ (and on p for the case of C_1) such that*

$$\begin{aligned} \|((N_\lambda \circ \mathcal{T}_\varphi^\mu)f)^{1/\rho}\|_{L^p(\mu)} &\leq \frac{C_1}{\lambda} \|f\|_{L^p(\mu)} \quad \text{and} \\ \mu(\{x \in \Gamma : (N_\lambda \circ \mathcal{T}_\varphi^\mu)f(x) > m\}) &\leq \frac{C_2}{\lambda m^{1/\rho}} \|f\|_{L^1(\mu)}. \end{aligned}$$

Trivially, $(N_a^b \circ \mathcal{T}_\varphi^\mu)f \leq (N_{b-a} \circ \mathcal{T}_\varphi^\mu)f$, thus Theorem 1.4 also holds replacing λ by $b - a$ and N_λ by N_a^b . In [JSW] it is shown that the results of Theorem 1.4 still hold when $\rho = 2$ for the particular case of the Hilbert transform. In our paper we do not pursue this endpoint result.

2. PRELIMINARIES

Throughout all the paper, n and d are two fixed integers such that $0 < n < d$. Given a point $x = (x^1, \dots, x^d) \in \mathbb{R}^d$, we use the notation $\tilde{x} := (x^1, \dots, x^n) \in \mathbb{R}^n$. Given a function $f : \mathbb{R}^m \rightarrow \mathbb{R}$, we denote by ∇f its gradient (when it makes sense), and by $\nabla^2 f$ the matrix of second derivatives of f . If f depends on different points $x_1, x_2, \dots \in \mathbb{R}^m$, then $\nabla_{x_i} f$ denotes the gradient of f with respect to the x_i variable, and analogously for $\nabla_{x_i}^2 f$.

For two sets $F_1, F_2 \subset \mathbb{R}^d$, we denote by $\text{dist}_{\mathcal{H}}(F_1, F_2)$ the Hausdorff distance between F_1 and F_2 . We denote by \mathcal{L}^n the Lebesgue measure on \mathbb{R}^n , and for the sake of simplicity, we set $\|\cdot\|_p := \|\cdot\|_{L^p(\mathcal{L}^n)}$ for $1 \leq p \leq \infty$, and $dy := d\mathcal{L}^n(y)$ for $y \in \mathbb{R}^n$.

In the paper, when we refer to the angle between two affine n -planes in \mathbb{R}^d , we mean the angle between the n -dimensional subspaces associated to the n -planes. As usual, the letter ‘ C ’ stands for some constant which may change its value at different occurrences, and which quite often only depends on n and d . The notation $A \lesssim B$ ($A \gtrsim B$) means that there is some fixed constant C such that $A \leq CB$ ($A \geq CB$), with C as above. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

2.1. More about the family φ . Given $x \in \mathbb{R}^d$, $0 < \epsilon \leq \delta$, and a finite Borel measure μ , we set $\varphi_\epsilon^\delta(x) := \varphi_\epsilon(x) - \varphi_\delta(x)$ and we define

$$(K\varphi_\epsilon^\delta * \mu)(x) := \int \varphi_\epsilon^\delta(x - y)K(x - y) d\mu(y),$$

thus $(K\varphi_\epsilon^\delta * \mu)(x) = (K\varphi_\epsilon * \mu)(x) - (K\varphi_\delta * \mu)(x)$.

For $m \in \mathbb{N}$, $x \in \mathbb{R}^m$, and $R \geq r > 0$, we denote by $B^m(x, r)$ the closed ball of \mathbb{R}^m with center x and radius r , and by $A^m(x, r, R)$ the closed annulus of \mathbb{R}^m centered at x with inner radius r and outer radius R . We also use the notation $B(x, r)$ and $A(x, r, R)$ when there is no possible confusion about m .

Each function φ_ϵ^δ is non negative, and $\text{supp}\varphi_\epsilon^\delta \subset A^n(0, 2.1\epsilon\sqrt{n}, 3\delta\sqrt{n}) \times \mathbb{R}^{d-n} \subset \mathbb{R}^d$. Moreover, $\sum_{j \in \mathbb{Z}} \varphi_{2^{-j}\epsilon}^{2^{-j}\delta}(\tilde{x}) = 1$ for $\tilde{x} \neq 0$, and there are at most two terms that do not vanish in the previous sum for a given $x \in \mathbb{R}^d$.

2.2. The α and β coefficients. Special dyadic lattice. Given $m \in \mathbb{N}$, $\lambda > 0$, and a cube $Q \subset \mathbb{R}^m$ (i.e. $Q := [0, b)^m + a$ with $a \in \mathbb{R}^m$ and $b > 0$), $\ell(Q)$ denotes the side length of Q , z_Q denotes the center of Q and λQ denotes the cube with center z_Q and side length $\lambda\ell(Q)$. Throughout the paper, we will only use cubes with sides parallel to the axes.

Let μ be a locally finite Borel measure on \mathbb{R}^d . Given $1 \leq p < \infty$ and a cube $Q \subset \mathbb{R}^d$, one sets (see [DS2])

$$(2.1) \quad \beta_{p,\mu}(Q) = \inf_L \left\{ \frac{1}{\ell(Q)^n} \int_{2Q} \left(\frac{\text{dist}(y, L)}{\ell(Q)} \right)^p d\mu(y) \right\}^{1/p},$$

where the infimum is taken over all n -planes L in \mathbb{R}^d . For $p = \infty$ one replaces the L^p norm by the supremum norm:

$$(2.2) \quad \beta_{\infty,\mu}(Q) = \inf_L \left\{ \sup_{y \in \text{supp}\mu \cap 2Q} \frac{\text{dist}(y, L)}{\ell(Q)} \right\},$$

where the infimum is taken over all n -planes L in \mathbb{R}^d again. These coefficients were introduced by P. W. Jones in [Jn1] for $p = \infty$ and by G. David and S. Semmes in [DS1] for $1 \leq p < \infty$.

Let $F \subset \mathbb{R}^d$ be the closure of an open set. Given two finite Borel measures σ, ν on \mathbb{R}^d , one sets

$$(2.3) \quad \text{dist}_F(\sigma, \nu) := \sup \left\{ \left| \int f d\sigma - \int f d\nu \right| : \text{Lip}(f) \leq 1, \text{supp} f \subset F \right\}.$$

It is easy to check that this is a distance in the space of finite Borel measures σ such that $\text{supp} \sigma \subset F$ and $\sigma(\partial F) = 0$. Moreover, it turns out that this distance is a variant of the well known Wasserstein distance W_1 from optimal transportation (see [Vi, Chapter 1]). See [Ma, Chapter 14] for other properties of dist_F .

Given a cube Q which intersects $\text{supp} \mu$, consider the closed ball $B_Q := B(z_Q, 6\ell(Q))$. Then one defines (see [To])

$$(2.4) \quad \alpha_\mu^n(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c \geq 0, L} \text{dist}_{B_Q}(\mu, c\mathcal{H}_L^n),$$

where the infimum is taken over all constants $c \geq 0$ and all n -planes L in \mathbb{R}^d . For convenience, if Q does not intersect $\text{supp} \mu$, we set $\alpha_\mu^n(Q) = 0$. To simplify notation, sometimes we will write $\alpha_\mu(Q)$ or $\alpha(Q)$ instead of $\alpha_\mu^n(Q)$ (and analogously for the β 's).

The following result characterizes uniform rectifiability in terms of the α and β coefficients.

Theorem 2.1. *Let μ be an n -dimensional AD regular measure on \mathbb{R}^d , and consider any $p \in [1, 2]$. Then, the following are equivalent:*

- (a) μ is uniformly n -rectifiable.
- (b) For any cube $R \subset \mathbb{R}^d$,

$$(2.5) \quad \sum_{Q \in \mathcal{D}_{\mathbb{R}^d}(R)} \beta_{p,\mu}(Q)^2 \ell(Q)^n \leq C \ell(R)^n$$

with C independent of R , where $\mathcal{D}_{\mathbb{R}^d}(R)$ stands for the collection of cubes of \mathbb{R}^d contained in R which are obtained by splitting R dyadically.

- (c) There exists $C > 0$ such that, for any cube $R \subset \mathbb{R}^d$,

$$(2.6) \quad \sum_{Q \in \mathcal{D}_{\mathbb{R}^d}(R)} \alpha_\mu(Q)^2 \ell(Q)^n \leq C \ell(R)^n.$$

The equivalence (a) \iff (b) in Theorem 2.1 was proved by G. David and S. Semmes in [DS1], and the equivalence (a) \iff (c) was proved by X. Tolsa in [To].

In this paper we will use a slightly different definition of the α and β coefficients adapted to the n -uniformly rectifiable measure $\mu = f\mathcal{H}_\Gamma^n$, where $\Gamma := \{x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x}))\}$ is the n -dimensional graph of a given Lipschitz function $A : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$ and $f \in L^\infty(\mathcal{H}_\Gamma^n)$ satisfies $f(x) \approx 1$ for almost all $x \in \Gamma$. To this end, we need to introduce a special dyadic lattice of sets related to Γ . Given a cube $\tilde{Q} \subset \mathbb{R}^n$ (i.e. $\tilde{Q} := [0, b)^n + a$ with $a \in \mathbb{R}^n$ and $b > 0$), we define $Q := \tilde{Q} \times \mathbb{R}^{d-n}$. This type of set will be called *v-cube* (“vertical” cube). We denote by $\ell(Q)$ and \tilde{z}_Q the side length and center of \tilde{Q} , respectively, and given $\lambda > 0$ we set $\lambda Q := \lambda \tilde{Q} \times \mathbb{R}^{d-n}$. Let $\tilde{\mathcal{D}}$ denote the standard dyadic lattice of \mathbb{R}^n , and set $\mathcal{D} := \{Q : \tilde{Q} \in \tilde{\mathcal{D}}\}$. It is easy to check that the v-cubes of \mathcal{D} intersected with Γ provide a dyadic lattice associated to the graph Γ in the sense of [Da, Appendix 1]. Finally, for $m \in \mathbb{Z}$, set $\mathcal{D}_m := \{Q \in \mathcal{D} : \ell(Q) = 2^{-m}\}$.

Fix a constant $C_\Gamma > 10\sqrt{n}(1 + \text{Lip}(A))$ (the precise value of C_Γ will not be relevant in the proofs given in the paper). Given $1 \leq p \leq \infty$ and a v-cube $Q \subset \mathbb{R}^d$, we define the coefficient

$\beta_{p,\mu}(Q)$ as in (2.1) and (2.2) but replacing $2Q$ by $C_\Gamma Q$. We also define $\alpha_\mu(Q)$ as in (2.4) but taking $B_Q := B(\tilde{z}_Q, C_\Gamma \ell(Q)) \times \mathbb{R}^{d-n} \subset \mathbb{R}^d$. This new definition of the α and β coefficients (adapted to the graph Γ) is the one that we will use in the whole paper.

Remark 2.2. It is an exercise to check that, with this new definition of the α 's and β 's, inequalities (2.5) and (2.6) of Theorem 2.1 still hold. Moreover, the following is an easy consequence of (2.5) and (2.6): *Let Γ be an n -dimensional Lipschitz graph, $f \in L^\infty(\mathcal{H}_\Gamma^n)$ such that $f(x) \approx 1$ for almost all $x \in \Gamma$, and $\mu = f\mathcal{H}_\Gamma^n$. Let $1 \leq p \leq 2$. Given $C_1, C_2, C_3 \geq 1$, there exists a constant $C_4 > 0$ such that, for any $R \in \mathcal{D}$,*

$$\sum_{Q \in \mathcal{D}: Q \subset C_1 R} (\beta_{p,\mu}(C_2 Q)^2 + \alpha_\mu(C_3 Q)^2) \mu(Q) \leq C_4 \mu(R),$$

and the dependence of C_4 with respect to Γ is only on $\text{Lip}(A)$.

Remark 2.3. It is shown in [To, Lemma 3.2], that $\beta_{1,\mu}(Q) \lesssim \alpha_\mu(Q)$ for all $Q \in \mathcal{D}$. Given $Q \in \mathcal{D}$, let L_Q be a minimizing n -plane for $\alpha_\mu(Q)$. In general, $\beta_{\infty,\mu}(Q)$ can not be controlled by $\beta_{1,\mu}(Q)$, so given $x \in \text{supp} \mu \cap C_\Gamma Q$, we can not control $\text{dist}(x, L_Q)$ by means of $\alpha_\mu(Q)$. But it is shown in [To, Lemma 5.2] that $\text{dist}(x, L_Q) \lesssim \sum_{R \in \mathcal{D}: x \in R \subset Q} \alpha_\mu(R) \ell(R)$, and in particular, if $P \in \mathcal{D}$ is such that $P \subset Q$ and $x \in \text{supp} \mu \cap C_\Gamma P$, and L_P denotes a minimizing n -plane for $\alpha_\mu(P)$, one has (see [To, Remark 5.3])

$$(2.7) \quad \text{dist}(x, L_Q) \lesssim \text{dist}(x, L_P) + \sum_{R \in \mathcal{D}: P \subset R \subset Q} \alpha_\mu(R) \ell(R).$$

2.3. Martingales. First of all, let us recall a particular case of Lépingle's inequality (see [JSW], or [Lé] and [JKRW, Theorem 6.4] for martingales in a probability space):

Theorem 2.4. *Let (X, Σ, λ) be a σ -finite measure space and $\rho > 2$. Then, there exist constants $C_1, C_2 > 0$ such that, for every martingale $\mathcal{G} := \{G_m\}_{m \in \mathbb{Z}} \in L^2(\lambda)$,*

$$\|\mathcal{V}_\rho(\mathcal{G})\|_{L^2(\lambda)} \leq C_1 \|\mathcal{G}\|_{L^2(\lambda)} \quad \text{and} \quad \|\mathcal{O}(\mathcal{G})\|_{L^2(\lambda)} \leq C_2 \|\mathcal{G}\|_{L^2(\lambda)},$$

where $\|\mathcal{G}\|_{L^2(\lambda)} := \sup_{m \in \mathbb{Z}} \|G_m\|_{L^2(\lambda)}$. The constants C_1 and C_2 do not depend on the measure λ , and C_2 neither depends on the fixed sequence that defines \mathcal{O} .

To prove Theorem 1.1, we need to introduce a particular martingale, and to review some known results.

Lemma 2.5. *Fix a cube $\tilde{P} \subset \mathbb{R}^n$ (not necessarily dyadic) and a Lipschitz graph $\Gamma := \{x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x}))\}$ such that $\text{supp} A \subset \tilde{P}$. Consider the measure $\mu := f\mathcal{H}_\Gamma^n$, where $f(x) = 1$ for all $\tilde{x} \in \tilde{P}^c$ and $C_0^{-1} \leq f(x) \leq C_0$ for all $\tilde{x} \in \tilde{P}$, for some fixed constant $C_0 > 0$. Also set $P := \tilde{P} \times \mathbb{R}^{d-n}$. Then, the following hold:*

$$(2.8) \quad T_* \mu \in L_{loc}^1(\mu), \quad T_*(\chi_E \mu) \in L_{loc}^1(\mu) \text{ for every compact set } E \subset \mathbb{R}^d, \text{ and}$$

$$(2.9) \quad \|T\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}.$$

Remark 2.6. To avoid the problem of non-integrability near infinity, for this type of measures μ we redefine $T_\epsilon \mu(x) := \lim_{M \rightarrow \infty} \int \chi_{(\epsilon, M)}(|x - y|) K(x - y) d\mu(y)$, which exists because μ is flat outside a compact set and K is odd. All the results in this paper remain valid with this new definition and the adjustments that have to be done in the proofs are minimal.

In this paper, we will deal with other integrals which concern the kernel K and the measure μ near infinity. The non-integrability problem can be avoided in the same manner.

Proof of Lemma 2.5. It is known that the operator T_*^μ is bounded in $L^2(\mu)$, because T_*^μ is the maximal operator associated to a Calderón-Zygmund singular integral and μ is a uniformly rectifiable measure (see [DS1]). Thus, $T_*(\chi_E\mu) = T_*^\mu(\chi_E) \in L_{loc}^1(\mu)$ for every compact set $E \subset \mathbb{R}^d$.

We are going to check that $\|T_*\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}$. This will imply that $T_*\mu \in L_{loc}^1(\mu)$ and, since $T\mu$ exists (because μ is uniformly rectifiable) and $|T\mu| \leq T_*\mu$, we will also obtain $\|T\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}$; so the lemma will be proved.

Using that T_*^μ is bounded in $L^2(\mu)$, we have

$$(2.10) \quad \begin{aligned} \|T_*\mu\|_{L^2(\mu)} &\leq \|T_*(\chi_{3P}\mu)\|_{L^2(\mu)} + \|T_*(\chi_{(3P)^c}\mu)\|_{L^2(\mu)} \\ &\lesssim \mu(P)^{1/2} + \|T_*(\chi_{(3P)^c}\mu)\|_{L^2(\mu)}. \end{aligned}$$

Set $L := \mathbb{R}^n \times \{0\}^{d-n} \subset \mathbb{R}^d$; obviously $\chi_{P^c}\mu = \mathcal{H}_{L \setminus P}^n$. Since L is an n -plane and K is odd, $T_*\mathcal{H}_L^n(x) = 0$ for all $x \in L$. Thus,

$$(2.11) \quad \|T_*\mathcal{H}_{L \setminus 3P}^n\|_{L^2(\mathcal{H}_L^n)} \leq \|T_*\mathcal{H}_L^n\|_{L^2(\mathcal{H}_L^n)} + \|T_*\mathcal{H}_{L \cap 3P}^n\|_{L^2(\mathcal{H}_L^n)} \lesssim \mu(P)^{1/2}.$$

Set $z_P := (\tilde{z}_P, 0, \dots, 0) \in L$ (recall that \tilde{z}_P denotes the center of \tilde{P}) and $\chi_\epsilon(x) := \chi_{(\epsilon, \infty)}(|x|)$. It is obvious that $\int \chi_\epsilon(z_P - y)K(z_P - y) d\mathcal{H}_{L \setminus 3P}^n(y) = 0$ for all $\epsilon > 0$. Thus, given $x \in \text{supp}\mu \cap P$,

$$\begin{aligned} |(K\chi_\epsilon * \mathcal{H}_{L \setminus 3P}^n)(x)| &\leq \int \chi_\epsilon(x - y)|K(x - y) - K(z_P - y)| d\mathcal{H}_{L \setminus 3P}^n(y) \\ &\quad + \int |\chi_\epsilon(x - y) - \chi_\epsilon(z_P - y)||K(z_P - y)| d\mathcal{H}_{L \setminus 3P}^n(y). \end{aligned}$$

Since Γ is a Lipschitz graph, $|x - z_P| \lesssim \ell(P)$. So, the first term on right hand side of the previous inequality is easily bounded by an absolute constant independent of ϵ , by standard arguments. For the second term, notice that $\text{supp}(\chi_\epsilon(x - \cdot) - \chi_\epsilon(z_P - \cdot)) \cap (L \setminus 3P) = \emptyset$ for all $\epsilon < \ell(P)$, and $\mathcal{H}_L^n(\{y \in \mathbb{R}^n : \chi_\epsilon(x - y) - \chi_\epsilon(z_P - y) \neq 0\}) \lesssim \ell(P)\epsilon^{n-1}$ for all $\epsilon \geq \ell(P)$. Therefore, since $|z_P - y| \approx \epsilon$ for all $y \in \text{supp}(\chi_\epsilon(x - \cdot) - \chi_\epsilon(z_P - \cdot)) \cap (L \setminus 3P)$, the second term can also be estimated by an absolute constant. Thus, we conclude $T_*\mathcal{H}_{L \setminus 3P}^n(x) = \sup_{\epsilon > 0} |(K\chi_\epsilon * \mathcal{H}_{L \setminus 3P}^n)(x)| \lesssim 1$ for all $x \in \text{supp}\mu \cap P$.

Using the previous observations and (2.11), we have

$$\begin{aligned} \|T_*(\chi_{(3P)^c}\mu)\|_{L^2(\mu)}^2 &= \|T_*\mathcal{H}_{L \setminus 3P}^n\|_{L^2(\chi_P\mu)}^2 + \|T_*\mathcal{H}_{L \setminus 3P}^n\|_{L^2(\chi_{P^c}\mu)}^2 \\ &\leq \|T_*\mathcal{H}_{L \setminus 3P}^n\|_{L^2(\chi_P\mu)}^2 + \|T_*\mathcal{H}_{L \setminus 3P}^n\|_{L^2(\mathcal{H}_L^n)}^2 \lesssim \mu(P), \end{aligned}$$

which, combined with (2.10), gives $\|T_*\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}$, as desired. \square

We are ready to define the martingale. Let P and μ be as in Lemma 2.5. Given $m \in \mathbb{Z}$ and $a \in \mathbb{R}^n$, we set

$$\tilde{D}_m^a := a + [0, 2^{-m}]^n \subset \mathbb{R}^n \quad \text{and} \quad D_m^a := \tilde{D}_m^a \times \mathbb{R}^{d-n} \subset \mathbb{R}^d.$$

Set $\mathcal{D}_m^a := \{\tilde{D}_m^{a+2^{-m}k} \subset \mathbb{R}^d : k \in \mathbb{Z}^n\}$ (notice that \mathcal{D}_m^a coincides with \mathcal{D}_m translated by a parameter $a \in \mathbb{R}^n$ and, for a fixed a , $\bigcup_{m \in \mathbb{Z}} \mathcal{D}_m^a$ is a translation of the standard dyadic lattice). Notice that $\mu(D_m^a) \approx 2^{-mn}$ for all $m \in \mathbb{Z}$, $a \in \mathbb{R}^n$. For $D \in \mathcal{D}_m^a$ and $x \in D$, we set

$$E_D\mu(x) := \frac{1}{\mu(D)} \int_D \int_{D^c} K(z - y) d\mu(y) d\mu(z)$$

(take into account Remark 2.6 for the meaning of $\int_{D^c} K(z - y) d\mu(y)$). Finally, for $x \in \mathbb{R}^d$, we define the martingale $E_m^a\mu(x) := \sum_{D \in \mathcal{D}_m^a} \chi_D(x) E_D\mu(x)$, $m \in \mathbb{Z}$.

Let us make some comments to understand better the nature of $E_m^a \mu$. First of all notice that, since $\mu(\partial D) = 0$, for any $D \in \mathcal{D}_m^a$ and μ -almost all $z \in D$ we have

$$(2.12) \quad \int_{D^c} K(z-y) d\mu(y) = \lim_{\epsilon \rightarrow 0} \int_{D^c} \chi_\epsilon(z-y) K(z-y) d\mu(y),$$

and for any $\epsilon > 0$, we have

$$(2.13) \quad \int_D \int_D \chi_\epsilon(z-y) K(z-y) d\mu(y) d\mu(z) = 0$$

because of the antisymmetry of K . Therefore, by (2.12), (2.13), (2.8), and the dominated convergence theorem, $\int_D |\int_{D^c} K(z-y) d\mu(y)| d\mu(z) < \infty$ (in particular, we have seen that $E_m^a \mu$ is well defined) and $\int_D T(\chi_D \mu) d\mu = 0$. Using this and (2.12), we finally have that

$$(2.14) \quad E_m^a \mu(x) = \frac{1}{\mu(D)} \int_D T(\chi_{D^c} \mu) d\mu = \frac{1}{\mu(D)} \int_D T\mu d\mu$$

for $x \in D \in \mathcal{D}_m^a$, thus $E_m^a \mu(x)$ is the average of the function $T\mu$ on the v -cube $D \in \mathcal{D}_m^a$ which contains x . So, it is completely clear that, for a fixed $a \in \mathbb{R}^n$, $\{E_m^a \mu\}_{m \in \mathbb{Z}}$ is a martingale. In [MV] it is shown that $\{E_m^a \mu\}_{m \in \mathbb{Z}}$ is well defined and it is a martingale without the assumption of the existence of $T\mu$ (i.e., for more general measures μ).

Now, we can use (2.14), the L^2 boundedness of the dyadic maximal operator and (2.9) to deduce that

$$(2.15) \quad \|E_m^a \mu\|_{L^2(\mu)} \lesssim \|T\mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}$$

for all $a \in \mathbb{R}^n$ and $m \in \mathbb{Z}$, where the constants that appear in the previous inequalities only depend on C_0 , n , d and $\text{Lip}(A)$.

Set $E^a \mu := \{E_m^a \mu\}_{m \in \mathbb{Z}}$. Then, the martingale $E^a \mu$ belongs to $L^2(\mu)$ by (2.15); thus by Theorem 2.4, for all $a \in \mathbb{R}^n$,

$$(2.16) \quad \begin{aligned} \|\mathcal{V}_\rho(E^a \mu)\|_{L^2(\mu)} &\lesssim \|E^a \mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2} \quad \text{for } \rho > 2, \\ \|\mathcal{O}(E^a \mu)\|_{L^2(\mu)} &\lesssim \|E^a \mu\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}, \end{aligned}$$

where the constants in the previous inequalities only depend on C_0 , n , d , and $\text{Lip}(A)$ (and on ρ , in the case of \mathcal{V}_ρ).

Finally, for $x \in \mathbb{R}^d$, we define

$$E_m \mu(x) := 2^{mn} \int_{\{a : x \in D_m^a\}} E_m^a \mu(x) da$$

(notice that $\mathcal{L}^n(\{a : x \in D_m^a\}) = 2^{-mn}$). Thus, $E_m \mu$ is an average (of the m 'th term) of some martingales depending on a parameter $a \in \mathbb{R}^n$.

Set $E\mu := \{E_m \mu\}_{m \in \mathbb{Z}}$. We want to obtain estimates like (2.16) for $\mathcal{V}_\rho(E\mu)$ and $\mathcal{O}(E\mu)$. We will only show the details for $\mathcal{V}_\rho(E\mu)$, because the case of $\mathcal{O}(E\mu)$ follows by similar arguments.

One can easily check that $E_m \mu(x) = 2^{Mn} \int_{[0, 2^{-M}]^n} E_m^a \mu(x) da$ for all $m, M \in \mathbb{Z}$ with $M \leq m$. Therefore, for all $M, r, s \in \mathbb{Z}$ with $M \leq r \leq s$, we have

$$(2.17) \quad E_r \mu(x) - E_s \mu(x) = 2^{Mn} \int_{[0, 2^{-M}]^n} (E_r^a \mu(x) - E_s^a \mu(x)) da.$$

Given $M \in \mathbb{Z}$, we consider the auxiliary transformation

$$\mathcal{V}_{\rho, M}(E\mu)(x) := \sup_{\{r_m\}} \left(\sum_{m \in \mathbb{Z}} |E_{r_{m+1}} \mu(x) - E_{r_m} \mu(x)|^\rho \right)^{1/\rho},$$

where the pointwise supremum is taken over all decreasing sequences of integers $\{r_m\}_{m \in \mathbb{Z}}$ such that $r_m \geq M$ for all $m \in \mathbb{Z}$. With this definition it is obvious that the sequence $\{\mathcal{V}_{\rho,M}(E\mu)(x)\}_{M \in \mathbb{Z}}$ is non increasing and $\mathcal{V}_\rho(E\mu)(x) = \lim_{M \rightarrow -\infty} \mathcal{V}_{\rho,M}(E\mu)(x)$ for all $x \in \mathbb{R}^d$. Minkowski's integral inequality and (2.17) yield the pointwise estimate

$$\begin{aligned} \mathcal{V}_{\rho,M}(E\mu)(x) &= \sup_{\{r_m\} : r_m \geq M} \left(\sum_{m \in \mathbb{Z}} |E_{r_{m+1}}\mu(x) - E_{r_m}\mu(x)|^\rho \right)^{1/\rho} \\ &\leq 2^{Mn} \int_{[0, 2^{-M}]^n} \sup_{\{r_m\}} \left(\sum_{m \in \mathbb{Z}} |E_{r_{m+1}}^a\mu(x) - E_{r_m}^a\mu(x)|^\rho \right)^{1/\rho} da \\ &= 2^{Mn} \int_{[0, 2^{-M}]^n} \mathcal{V}_\rho(E^a\mu)(x) da. \end{aligned}$$

Therefore, by the previous estimate, Minkowski's integral inequality and (2.16),

$$\|\mathcal{V}_{\rho,M}(E\mu)\|_{L^2(\mu)} \leq 2^{Mn} \int_{[0, 2^{-M}]^n} \|\mathcal{V}_\rho(E^a\mu)\|_{L^2(\mu)} da \leq C\mu(P)^{1/2},$$

where $C > 0$ only depends on C_0 , n , d , $\text{Lip}(A)$, and ρ . By the monotone convergence theorem, we conclude that $\|\mathcal{V}_\rho(E\mu)\|_{L^2(\mu)} \lesssim \mu(P)^{1/2}$. Thus we have proved the following theorem (which can be considered the starting point to prove Theorem 1.1):

Theorem 2.7. *Fix a cube $\tilde{P} \subset \mathbb{R}^n$. Set $\Gamma := \{x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x}))\}$, where $A : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$ is a Lipschitz function supported in \tilde{P} , and set $P := \tilde{P} \times \mathbb{R}^{d-n}$. Set $\mu := f\mathcal{H}_\Gamma^n$, where $f(x) = 1$ for all $\tilde{x} \in \tilde{P}^c$ and $C_0^{-1} \leq f(x) \leq C_0$ for all $\tilde{x} \in \tilde{P}$, for some constant $C_0 > 0$.*

Let $\rho > 2$. Then, there exist constants $C_1, C_2 > 0$ such that $\|\mathcal{V}_\rho(E\mu)\|_{L^2(\mu)} \leq C_1\mu(P)^{1/2}$ and $\|\mathcal{O}(E\mu)\|_{L^2(\mu)} \leq C_2\mu(P)^{1/2}$, where C_1 and C_2 only depend on C_0 , n , d , and $\text{Lip}(A)$ (and on ρ in the case of C_1).

We need to introduce additional notation in order to express $E_m\mu$ in a more convenient way for our purposes. Let μ_1, \dots, μ_k be a finite collection of positive Borel measures such that $\mu_l(D_m^a) > 0$ for all $a \in \mathbb{R}^n$, $m \in \mathbb{Z}$ and $l = 1, \dots, k$. Given $m \in \mathbb{Z}$ and $x_1, \dots, x_i, y_1, \dots, y_j \in \mathbb{R}^d$, we define

$$\Lambda_m^{\mu_1, \dots, \mu_k}(x_1, \dots, x_i; y_1, \dots, y_j) := 2^{nm} \int_{\{a : x_1, \dots, x_i \in D_m^a, y_1, \dots, y_j \notin D_m^a\}} \frac{da}{\prod_{l=1}^k \mu_l(D_m^a)}.$$

Then, by Fubini's theorem,

$$\begin{aligned} E_m\mu(x) &= \int_{\{a : x \in D_m^a\}} \frac{2^{mn}}{\mu(D_m^a)} \int_{D_m^a} \int_{(D_m^a)^c} K(z - y) d\mu(y) d\mu(z) da \\ (2.18) \quad &= \iint \left(2^{mn} \int_{\{a : x, z \in D_m^a, y \notin D_m^a\}} \frac{da}{\mu(D_m^a)} \right) K(z - y) d\mu(z) d\mu(y) \\ &= \iint \Lambda_m^\mu(x, z; y) K(z - y) d\mu(z) d\mu(y). \end{aligned}$$

3. SKETCH OF THE PROOF OF THEOREM 1.1

The proof relies on two basic facts: the known L^2 boundedness of the ρ -variation and oscillation of martingales explained in the previous section and the good geometric properties of Lipschitz graphs from a measure-theoretic point of view.

As we said above, the starting point of the proof is Theorem 2.7, where the L^2 boundedness of the ρ -variation and oscillation (of a convex combination) of some particular martingales is stated. So, the first step consists in relating the results on martingales in Theorem 2.7 with the ρ -variation and oscillation of singular integrals on Lipschitz graphs, and this is the aim of the following two theorems:

Theorem 3.1. *Let Γ and μ be as in Theorem 2.7. For each $x \in \Gamma$, define*

$$(3.1) \quad W\mu(x)^2 := \sum_{m \in \mathbb{Z}} |(K\varphi_{2^{-m}} * \mu)(x) - E_m\mu(x)|^2.$$

Then, $\|W\mu\|_{L^2(\mu)}^2 \leq C_1 \sum_{Q \in \mathcal{D}} (\alpha_\mu(C_2 Q)^2 + \beta_{2,\mu}(Q)^2) \mu(Q)$, where $C_1, C_2 > 0$ depend only on C_0, n, d, K , and $\text{Lip}(A)$.

Theorem 3.2. *Let Γ and μ be as in Theorem 2.7. For each $x \in \Gamma$, define*

$$(3.2) \quad S\mu(x)^2 := \sup_{\{\epsilon_m\}} \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}: \epsilon_m, \epsilon_{m+1} \in I_j} |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^2,$$

where $I_j = [2^{-j-1}, 2^{-j})$ and the supremum is taken over all decreasing sequences of positive numbers $\{\epsilon_m\}_{m \in \mathbb{Z}}$. Then, $\|S\mu\|_{L^2(\mu)}^2 \leq C \sum_{Q \in \mathcal{D}} (\alpha_\mu(Q)^2 + \beta_{2,\mu}(Q)^2) \mu(Q)$, where $C > 0$ only depends on C_0, n, d, K , and $\text{Lip}(A)$.

Two fundamental tools to study $W\mu$ and $S\mu$ are the α and β coefficients, which will be used to measure the flatness of Γ at different scales, in order to estimate the terms which appear in the sums in (3.1) and (3.2). This will be done in sections 4 and 5. To use the α coefficients to relate the ρ -variation of martingales with the ρ -variation of singular integrals, it is a key fact that we are considering a “smooth” family like φ , because the α ’s are defined in terms of Lipschitz functions but T_ϵ is defined by means of a rough truncation. Moreover, we are taking a truncation only on the first n -coordinates because the average of martingales that we are using is taken over the parameter $a \in \mathbb{R}^n$, using the v-cubes D_M^a (see subsection 2.3).

Combining Theorem 3.1 and Theorem 3.2 with the L^2 estimates of the ρ -variation and oscillation on the average of martingales $E\mu$ in Theorem 2.7, we are able to obtain local L^2 estimates of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ when Γ is any Lipschitz graph. More precisely, we separate the sum in the definition of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ into two parts, which are classically called short and long variation (and analogously for $\mathcal{O} \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$). The short variation corresponds to the sum $S\mu$ in Theorem 3.2 (here μ is a suitable modification of \mathcal{H}_Γ^n), where the indices run over $m \in \mathbb{Z}$ such that both ϵ_m and ϵ_{m+1} lie in the same dyadic interval, and can be handled using the α ’s and β ’s. The long variation corresponds to the sum over the indices $m \in \mathbb{Z}$ such that ϵ_m and ϵ_{m+1} lie in different dyadic intervals, so one may assume that the ϵ_m ’s are dyadic numbers. It is handled by comparing $K\varphi_{2^{-m}} * \mu$ with $E_m\mu$, and then using Theorem 3.1 and the fact the ρ -variation and oscillation of $E\mu$ are bounded in $L^2(\mu)$, by Theorem 2.7. This will be done in section 6 (see Theorem 6.1).

Using the local L^2 estimates of Theorem 6.1, combined with rather standard techniques in Calderón-Zygmund theory, in section 7 we obtain the $H^1(\mathcal{H}_\Gamma^n) \rightarrow L^1(\mathcal{H}_\Gamma^n)$ and $L^\infty(\mathcal{H}_\Gamma^n) \rightarrow BMO(\mathcal{H}_\Gamma^n)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$. Then, by interpolation, we obtain the L^p boundedness of these operators in the whole range $1 < p < \infty$, and in particular the L^2 boundedness (see Theorem 7.1). Moreover, [CJRW2, Theorem B] can be adapted to prove that the $L^2(\mathcal{H}_\Gamma^n)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ also yields the boundedness of these operators from $L^1(\mathcal{H}_\Gamma^n)$ to $L^{1,\infty}(\mathcal{H}_\Gamma^n)$.

Let us stress that almost all the estimates in the proof of Theorem 1.1 (in particular, the constants involved in the relationships \lesssim , \gtrsim and \approx) depend either on n , d , K or $\text{Lip}(A)$, and possibly on other variables such as ρ or p .

4. PROOF OF THEOREM 3.1

In order to study the difference $(K\varphi_{2^{-m}}*\mu)(x) - E_m\mu(x)$, we are going to split $E_m\mu(x)$ into two parts, the one we will compare with $(K\varphi_{2^{-m}}*\mu)(x)$ (which corresponds to integrate, in the definition of $E_m\mu(x)$, over the points $y \in \mathbb{R}^d$ such that $2^{-m} \lesssim |\tilde{x} - \tilde{y}|$), and the remaining part. Then, we will estimate each part of $(K\varphi_{2^{-m}}*\mu)(x) - E_m\mu(x)$ separately, using the cancelation properties of the kernel K and the uniform rectifiability of μ .

Recall from (2.18) that $E_m\mu(x) = \iint \Lambda_m^\mu(x, z; y) K(z - y) d\mu(z) d\mu(y)$. Given $\epsilon > 0$, we set $\gamma_\epsilon := 1 - \varphi_\epsilon$. Then,

$$\begin{aligned} E_m\mu(x) &= \iint \varphi_{2^{-m}}(x - y) \Lambda_m^\mu(x, z; y) K(z - y) d\mu(z) d\mu(y) \\ &\quad + \iint \gamma_{2^{-m}}(x - y) \Lambda_m^\mu(x, z; y) K(z - y) d\mu(z) d\mu(y). \end{aligned}$$

The first term in the previous sum is the one that we will compare with $(K\varphi_{2^{-m}}*\mu)(x)$. For all $a \in \mathbb{R}^n$ such that $x \in D_m^a$, we have $\text{supp } \varphi_{2^{-m}}(x - \cdot) \cap D_m^a = \emptyset$, and thus $(K\varphi_{2^{-m}}*\mu)(x) = (K\varphi_{2^{-m}}*(\chi_{(D_m^a)^c}\mu))(x)$. Hence, using Fubini's theorem and the definition of $\Lambda_m^\mu(x, z; y)$,

$$\begin{aligned} (K\varphi_{2^{-m}}*\mu)(x) &= 2^{mn} \int_{\{a: x \in D_m^a\}} (K\varphi_{2^{-m}}*(\chi_{(D_m^a)^c}\mu))(x) da \\ &= 2^{mn} \int_{\{a: x \in D_m^a\}} \mu(D_m^a)^{-1} \int_{D_m^a} (K\varphi_{2^{-m}}*(\chi_{(D_m^a)^c}\mu))(x) d\mu(z) da \\ &= \iint \varphi_{2^{-m}}(x - y) \Lambda_m^\mu(x, z; y) K(x - y) d\mu(z) d\mu(y). \end{aligned}$$

We can decompose $(K\varphi_{2^{-m}}*\mu)(x) - E_m\mu(x)$ as

$$\begin{aligned} (4.1) \quad &(K\varphi_{2^{-m}}*\mu)(x) - E_m\mu(x) \\ &= \iint \varphi_{2^{-m}}(x - y) \Lambda_m^\mu(x, z; y) (K(x - y) - K(z - y)) d\mu(z) d\mu(y) \\ &\quad - \iint \gamma_{2^{-m}}(x - y) \Lambda_m^\mu(x, z; y) K(z - y) d\mu(z) d\mu(y) \\ &= \sum_{j < m} F_j^m(x) - \sum_{j \in \mathbb{Z}} G_j^m(x), \end{aligned}$$

where

$$(4.2) \quad F_j^m(x) := \iint \varphi_{2^{-j-1}}^{2^{-j}}(x - y) \Lambda_m^\mu(x, z; y) (K(x - y) - K(z - y)) d\mu(z) d\mu(y),$$

$$(4.3) \quad G_j^m(x) := \iint \varphi_{2^{-j-1}}^{2^{-j}}(z - y) \gamma_{2^{-m}}(x - y) \Lambda_m^\mu(x, z; y) K(z - y) d\mu(z) d\mu(y).$$

Fix a v -cube $D \in \mathcal{D}_m$, for some $m \in \mathbb{Z}$. In subsection 4.1 (see (4.18)) we will prove that

$$(4.4) \quad \sum_{j < m} |F_j^m(x)| \lesssim \frac{\text{dist}(x, L_D)}{\ell(D)} + \sum_{Q \in \mathcal{D}: D \subset Q} \frac{\ell(D)}{\ell(Q)} \alpha(Q)$$

for all $x \in D \cap \Gamma$, where L_D denotes an n -plane that minimizes $\alpha(D)$, and in subsection 4.2 (see (4.37)) we will prove that there exists a constant $C_b > 1$ such that

$$(4.5) \quad \sum_{j \in \mathbb{Z}} |G_j^m(x)| \lesssim \alpha(C_b D) + \sum_{Q \in \mathcal{D}: Q \subset C_b D} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}} \alpha(Q)$$

for all $x \in D \cap \Gamma$. Assuming that these estimates hold, by (4.1),

$$(4.6) \quad \begin{aligned} \|W\mu\|_{L^2(\mu)}^2 &= \sum_{m \in \mathbb{Z}} \sum_{D \in \mathcal{D}_m} \int_D |(K\varphi_{2^{-m}} * \mu)(x) - E_m \mu(x)|^2 d\mu(x) \\ &\lesssim \sum_{D \in \mathcal{D}} \int_D \left(\frac{\text{dist}(x, L_D)}{\ell(D)} \right)^2 d\mu(x) + \sum_{D \in \mathcal{D}} \left(\sum_{\substack{Q \in \mathcal{D}: \\ D \subset Q}} \frac{\ell(D)}{\ell(Q)} \alpha(Q) \right)^2 \mu(D) \\ &\quad + \sum_{D \in \mathcal{D}} \alpha(C_b D)^2 \mu(D) + \sum_{D \in \mathcal{D}} \left(\sum_{\substack{Q \in \mathcal{D}: \\ Q \subset C_b D}} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}} \alpha(Q) \right)^2 \mu(D) \\ &=: W_1\mu + W_2\mu + W_3\mu + W_4\mu. \end{aligned}$$

If L_D^1 and L_D^2 denote a minimizing n -plane for $\beta_1(D)$ and $\beta_2(D)$, respectively, one can show that $\text{dist}_{\mathcal{H}}(L_D \cap C_\Gamma D, L_D^1 \cap C_\Gamma D) \lesssim \alpha(D)\ell(D)$ and $\text{dist}_{\mathcal{H}}(L_D^1 \cap C_\Gamma D, L_D^2 \cap C_\Gamma D) \lesssim \beta_2(D)\ell(D)$. This easily implies that, for $x \in D \cap \Gamma$, $\text{dist}(x, L_D) \lesssim \text{dist}(x, L_D^2) + \beta_2(D)\ell(D) + \alpha(D)\ell(D)$, so $W_1\mu \lesssim \sum_{D \in \mathcal{D}} (\alpha(D)^2 + \beta_2(D)^2)\mu(D)$.

By Cauchy-Schwarz inequality,

$$\begin{aligned} W_2\mu &\leq \sum_{D \in \mathcal{D}} \mu(D) \left(\sum_{Q \in \mathcal{D}: D \subset Q} \frac{\ell(D)}{\ell(Q)} \alpha(Q)^2 \right) \left(\sum_{Q \in \mathcal{D}: D \subset Q} \frac{\ell(D)}{\ell(Q)} \right) \\ &\approx \sum_{D \in \mathcal{D}} \sum_{Q \in \mathcal{D}: D \subset Q} \frac{\ell(D)^{n+1}}{\ell(Q)} \alpha(Q)^2 \approx \sum_{Q \in \mathcal{D}} \alpha(Q)^2 \mu(Q), \end{aligned}$$

and also

$$\begin{aligned} W_4\mu &\leq \sum_{D \in \mathcal{D}} \mu(D) \left(\sum_{Q \in \mathcal{D}: Q \subset C_b D} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}} \alpha(Q)^2 \right) \left(\sum_{Q \in \mathcal{D}: Q \subset C_b D} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}} \right) \\ &\approx \sum_{D \in \mathcal{D}} \sum_{Q \in \mathcal{D}: Q \subset C_b D} \ell(Q)^n \frac{\ell(Q)}{\ell(D)} \alpha(Q)^2 \lesssim \sum_{Q \in \mathcal{D}} \alpha(Q)^2 \mu(Q). \end{aligned}$$

Therefore, using (4.6) and that $\alpha(Q) \lesssim \alpha(C_b Q)$, we conclude that

$$\|W\mu\|_{L^2(\mu)}^2 \lesssim \sum_{Q \in \mathcal{D}} (\alpha(C_b Q)^2 + \beta_2(Q)^2) \mu(Q),$$

and the theorem follows. It only remains to prove (4.4) and (4.5).

4.1. Estimate of $\sum_{j < m} F_j^m(x)$ when $x \in D \cap \Gamma$ for some $D \in \mathcal{D}_m$. Assume that $x \in D \cap \Gamma$ for some $D \in \mathcal{D}_m$ and $j < m$. Let L_D be an n -plane that minimizes $\alpha(D)$ and let $\sigma_D := c_D \mathcal{H}_{L_D}^n$ be a minimizing measure of $\alpha(D)$. Let L_D^x be the n -plane parallel to L_D that contains x and set $\sigma_D^x := c_D \mathcal{H}_{L_D^x}^n$.

Notice that, because of $x \in L_D^x$, the antisymmetry of $\varphi_{2^{-j-1}}^{2^{-j}}K$, and since $j < m$ (so, if $x \in D_m^a$ and $y \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(x - \cdot)$, then $y \notin D_m^a$), we have

$$\begin{aligned}
 0 &= \int \varphi_{2^{-j-1}}^{2^{-j}}(x - y)K(x - y) d\sigma_D^x(y) \\
 (4.7) \quad &= \int_{\{a: x \in D_m^a\}} \frac{2^{mn}}{\sigma_D^x(D_m^a)} \int_{D_m^a} \int_{(D_m^a)^c} \varphi_{2^{-j-1}}^{2^{-j}}(x - y)K(x - y) d\sigma_D^x(y) d\sigma_D^x(z) da \\
 &= \iint \varphi_{2^{-j-1}}^{2^{-j}}(x - y)\Lambda_m^{\sigma_D^x}(x, z; y)K(x - y) d\sigma_D^x(z) d\sigma_D^x(y).
 \end{aligned}$$

Given $a \in \mathbb{R}^n$, let $b := a + \{2^{-m-1}\}^n \in \mathbb{R}^n$ be the center of \tilde{D}_m^a . For $u \in \mathbb{R}^n$ we denote $\|u\|_\infty := \max_{i=1, \dots, n} |u^i|$. Then, given $t \in \mathbb{R}^d$, it is clear that $t \in \tilde{D}_m^a$ if and only if $\|\tilde{t} - b\|_\infty \leq 2^{-m}$. Using that σ_D^x is a Hausdorff measure on an n -plane, that K is antisymmetric and that $\varphi_{2^{-j-1}}^{2^{-j}}$ is symmetric, one can show that

$$0 = \int_{\|\tilde{x}-b\|_\infty \leq 2^{-m}} \int_{\|\tilde{z}-b\|_\infty \leq 2^{-m}} \int_{\|\tilde{y}-b\|_\infty > 2^{-m}} \varphi_{2^{-j-1}}^{2^{-j}}(x - y)K(z - y) d\sigma_D^x(y) d\sigma_D^x(z) db.$$

By the change of variable $b = a + \{2^{-m-1}\}^n$, it is easy to see that this triple integral is equal to $\int_{\{a: x \in D_m^a\}} \int_{D_m^a} \int_{(D_m^a)^c} \varphi_{2^{-j-1}}^{2^{-j}}(x - y)K(z - y) d\sigma_D^x(y) d\sigma_D^x(z) da$. Thus, since $\sigma_D^x(D_m^a)$ does not depend on $a \in \mathbb{R}^n$ because σ_D^x is flat,

$$\begin{aligned}
 0 &= \int_{\{a: x \in D_m^a\}} \frac{2^{mn}}{\sigma_D^x(D_m^a)} \int_{D_m^a} \int_{(D_m^a)^c} \varphi_{2^{-j-1}}^{2^{-j}}(x - y)K(z - y) d\sigma_D^x(y) d\sigma_D^x(z) da \\
 (4.8) \quad &= \iint \varphi_{2^{-j-1}}^{2^{-j}}(x - y)\Lambda_m^{\sigma_D^x}(x, z; y)K(z - y) d\sigma_D^x(z) d\sigma_D^x(y).
 \end{aligned}$$

By (4.7) and (4.8), we conclude that

$$(4.9) \quad 0 = \iint \varphi_{2^{-j-1}}^{2^{-j}}(x - y)\Lambda_m^{\sigma_D^x}(x, z; y)(K(x - y) - K(z - y)) d\sigma_D^x(z) d\sigma_D^x(y).$$

By definition, it is clear that $\Lambda_m^{\sigma_D^x}(x, z; y) = \Lambda_m^{\sigma_D}(x, z; y)$. Therefore, using (4.9), we can decompose

$$(4.10) \quad F_j^m(x) = F1_j^m(x) + F2_j^m(x) + F3_j^m(x) + F4_j^m(x),$$

where

$$(4.11) \quad F1_j^m(x) := \iint \varphi_{2^{-j-1}}^{2^{-j}}(x - y)\Lambda_m^\mu(x, z; y) (K(x - y) - K(z - y)) d(\mu - \sigma_D)(z) d\mu(y),$$

$$(4.12) \quad F2_j^m(x) := \iint \varphi_{2^{-j-1}}^{2^{-j}}(x - y)\Lambda_m^\mu(x, z; y) (K(x - y) - K(z - y)) d\sigma_D(z) d(\mu - \sigma_D)(y),$$

$$(4.13) \quad F3_j^m(x) := \iint \varphi_{2^{-j-1}}^{2^{-j}}(x - y)(\Lambda_m^\mu(x, z; y) - \Lambda_m^{\sigma_D}(x, z; y)) (K(x - y) - K(z - y)) d\sigma_D(z) d\sigma_D(y),$$

$$(4.14) \quad F4_j^m(x) := \iint \varphi_{2^{-j-1}}^{2^{-j}}(x - y)\Lambda_m^{\sigma_D^x}(x, z; y) (K(x - y) - K(z - y)) d(\sigma_D \times \sigma_D - \sigma_D^x \times \sigma_D^x)(z, y).$$

In the next subsections we will prove the following estimates:

$$(4.15) \quad |F1_j^m(x)| + |F3_j^m(x)| \lesssim 2^{j-m} \alpha(D),$$

$$(4.16) \quad |F2_j^m(x)| \lesssim 2^{j-m} \sum_{Q \in \mathcal{D}: D \subset Q, \ell(Q) \leq 2^{-j}} \alpha(Q),$$

$$(4.17) \quad |F4_j^m(x)| \lesssim 2^{j-m} \frac{\text{dist}(x, L_D)}{\ell(D)}.$$

Then, using (4.10), we will finally get that, for all $D \in \mathcal{D}_m$ and $x \in D \cap \Gamma$,

$$(4.18) \quad \begin{aligned} \sum_{j < m} |F_j^m(x)| &\lesssim \frac{\text{dist}(x, L_D)}{\ell(D)} + \sum_{j \leq m} 2^{j-m} \sum_{Q \in \mathcal{D}: D \subset Q, \ell(Q) \leq 2^{-j}} \alpha(Q) \\ &\lesssim \frac{\text{dist}(x, L_D)}{\ell(D)} + \sum_{Q \in \mathcal{D}: D \subset Q} \frac{\ell(D)}{\ell(Q)} \alpha(Q), \end{aligned}$$

which gives (4.4).

4.1.1. Estimate of $F1_j^m(x)$. Notice that, if $|\tilde{x} - \tilde{z}| > 2^{-m} \sqrt{n}$, there is no $a \in \mathbb{R}^n$ such that $x, z \in D_m^a$, and this means that $\Lambda_m^\mu(x, z; y) = 0$. Thus, we can assume that $|\tilde{x} - \tilde{z}| \leq 2^{-m} \sqrt{n}$. Therefore, if the constant C_Γ (see the definition of the α 's in subsection 2.2) is big enough, $\text{supp} \Lambda_m^\mu(x, \cdot; y) \subset B_D$.

For $y, z \in \Gamma$ such that $y \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(x - \cdot)$, $j < m$ and $|\tilde{x} - \tilde{z}| \leq 2^{-m} \sqrt{n}$ (so, in particular, $|x - z| \lesssim |x - y|$), we have the following estimates:

$$\begin{aligned} |K(x - y) - K(z - y)| &\lesssim |x - z| |x - y|^{-n-1} \lesssim 2^{j(n+1)-m}, \\ |\nabla_z(K(x - y) - K(z - y))| &= |\nabla_z K(z - y)| \lesssim 2^{j(n+1)}. \end{aligned}$$

Claim 4.1. *We have $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{mn}$ and $|\nabla_z \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$ for all $x, y, z \in \mathbb{R}^d$.*

Claim 4.1 and the subsequent ones 4.2, ..., 4.7 will be proved in subsection 4.3 below.

Putting all these estimates together we obtain that

$$\left| \nabla_z \left(\Lambda_m^\mu(x, z; y) (K(x - y) - K(z - y)) \right) \right| \lesssim 2^{j(n+1)+mn},$$

and, since $\text{supp} \Lambda_m^\mu(x, \cdot; y) \subset B_D$, recalling the definition of dist_{B_D} in (2.3),

$$\left| \int \Lambda_m^\mu(x, z; y) (K(x - y) - K(z - y)) d(\mu - \sigma_D)(z) \right| \lesssim 2^{j(n+1)+mn} \text{dist}_{B_D}(\mu, \sigma_D).$$

We can use this last estimate in (4.11) to obtain

$$\begin{aligned} |F1_j^m(x)| &\lesssim 2^{j(n+1)+mn} \text{dist}_{B_D}(\mu, \sigma_D) \int \varphi_{2^{-j-1}}^{2^{-j}}(x - y) d\mu(y) \\ &\lesssim 2^{j+mn} \text{dist}_{B_D}(\mu, \sigma_D) \approx 2^{j-m} \ell(D)^{-n-1} \text{dist}_{B_D}(\mu, \sigma_D) \lesssim 2^{j-m} \alpha(D), \end{aligned}$$

which, together with the estimate of $|F3_j^m(x)|$ in subsection 4.1.3, gives (4.15).

4.1.2. Estimate of $F2_j^m(x)$. Arguing as in subsection 4.1.1, we can obtain the following estimates for x, y, z as above:

$$(4.19) \quad |\varphi_{2^{-j-1}}^{2^{-j}}(x - y)| \leq 1 \quad \text{and} \quad |\nabla_y \varphi_{2^{-j-1}}^{2^{-j}}(x - y)| \lesssim 2^j,$$

$$(4.20) \quad |K(x - y) - K(z - y)| \lesssim |x - z| |x - y|^{-n-1} \lesssim 2^{j(n+1)-m},$$

$$(4.21) \quad |\nabla_y(K(x - y) - K(z - y))| \lesssim |x - z| |x - y|^{-n-2} \lesssim 2^{j(n+2)-m}.$$

Claim 4.2. *For $j < m$, $y \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(x - \cdot)$, and $|\tilde{x} - \tilde{z}| \leq 2^{-m} \sqrt{n}$, the following hold: $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{mn}$ and $\nabla_y \Lambda_m^\mu(x, z; y) = 0$.*

Notice that the first estimate in Claim 4.2 is the same as the first one in Claim 4.1.

Let $D_j \in \mathcal{D}_j$ be the unique dyadic v-cube with $\ell(D_j) = 2^{-j}$ which contains D . Then, $\text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(x - \cdot) \subset B_{D_j}$ for C_Γ big enough. Therefore, we can use the previous estimates to see that the gradient of the term inside the integral with respect to y in (4.12) is bounded by $2^{j(n+2)+m(n-1)}$ and is supported in B_{D_j} , and then by (2.3) we derive that

$$\begin{aligned}
 |F2_j^m(x)| &\leq \int \left| \int \varphi_{2^{-j-1}}^{2^{-j}}(x - y) \Lambda_m^\mu(x, z; y) \right. \\
 &\quad \left. (K(x - y) - K(z - y)) d(\mu - \sigma_D)(y) \right| d\sigma_D(z) \\
 (4.22) \quad &\lesssim \int_{|\tilde{x} - \tilde{z}| \leq 2^{-m} \sqrt{n}} 2^{j(n+2)+m(n-1)} \text{dist}_{B_{D_j}}(\mu, \sigma_D) d\sigma_D(z) \\
 &\lesssim 2^{j(n+2)-m} \text{dist}_{B_{D_j}}(\mu, \sigma_D).
 \end{aligned}$$

We shall estimate $\text{dist}_{B_{D_j}}(\mu, \sigma_D)$ in terms of the α coefficients. Consider the unique sequence of dyadic v-cubes $D =: D_m \subset \dots \subset D_{i+1} \subset D_i \subset \dots \subset D_j$ such that each D_i belongs to \mathcal{D}_i , for $i = j, \dots, m$. Let L_{D_i} be an n -plane that gives the minimum in the definition of $\alpha(D_i)$ and let $\sigma_{D_i} := c_{D_i} d\mathcal{H}_{L_{D_i}}^n$ be a minimizing measure. We will prove that

$$(4.23) \quad \text{dist}_{B_{D_j}}(\mu, \sigma_D) \lesssim 2^{-j(n+1)} \sum_{i=j}^{m-1} \alpha(D_i).$$

Combining (4.23) with (4.22), we will finally obtain that $|F2_j^m(x)| \lesssim 2^{j-m} \sum_{i=j}^{m-1} \alpha(D_i)$, which gives (4.16).

Let us prove (4.23). By the triangle inequality,

$$\begin{aligned}
 \text{dist}_{B_{D_j}}(\mu, \sigma_D) &\leq \text{dist}_{B_{D_j}}(\mu, \sigma_{D_j}) + \sum_{i=j}^{m-1} \text{dist}_{B_{D_j}}(\sigma_{D_i}, \sigma_{D_{i+1}}) \\
 &\lesssim 2^{-j(n+1)} \alpha(D_j) + \sum_{i=j}^{m-1} \text{dist}_{B_{D_j}}(\sigma_{D_i}, \sigma_{D_{i+1}}),
 \end{aligned}$$

so we are reduced to prove that, for all $i = j, \dots, m-1$,

$$(4.24) \quad \text{dist}_{B_{D_j}}(\sigma_{D_i}, \sigma_{D_{i+1}}) \lesssim 2^{-j(n+1)} \alpha(D_i).$$

By definition, $\text{dist}_{B_{D_j}}(\sigma_{D_i}, \sigma_{D_{i+1}}) = \sup \left| \int g d(c_{D_i} \mathcal{H}_{L_{D_i}}^n - c_{D_{i+1}} \mathcal{H}_{L_{D_{i+1}}}^n) \right|$, where the supremum is taken over all Lipschitz functions g supported in B_{D_j} such that $\text{Lip}(g) \leq 1$. Fix one of such Lipschitz functions g . Then,

$$\begin{aligned}
 (4.25) \quad \int g d(c_{D_i} \mathcal{H}_{L_{D_i}}^n - c_{D_{i+1}} \mathcal{H}_{L_{D_{i+1}}}^n) &= (c_{D_i} - c_{D_{i+1}}) \int g d\mathcal{H}_{L_{D_i}}^n \\
 &\quad + c_{D_{i+1}} \int g d(\mathcal{H}_{L_{D_i}}^n - \mathcal{H}_{L_{D_{i+1}}}^n).
 \end{aligned}$$

It is shown in [To, Lemma 3.4] that $|c_{D_i} - c_{D_{i+1}}| \lesssim \alpha(D_i)$, so the first term on the right hand side of (4.25) is bounded in absolute value by $C2^{-j(n+1)} \alpha(D_i)$.

In order to estimate the second term of the right hand side of (4.25), set $L_{D_{i+1}} = \{(\tilde{t}, a(\tilde{t})) \in \mathbb{R}^d : \tilde{t} \in \mathbb{R}^n\}$ (where $a : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$ is an appropriate affine map), and let $p : L_{D_i} \rightarrow L_{D_{i+1}}$ be the projection defined by $p(t) := (\tilde{t}, a(\tilde{t}))$. Since Γ is a Lipschitz graph,

a is well defined and p is a homeomorphism. Let $p_{\#}\mathcal{H}_{L_{D_i}}^n$ be the image measure of $\mathcal{H}_{L_{D_i}}^n$ by p . It is easy to check that $\mathcal{H}_{L_{D_{i+1}}}^n = \tau p_{\#}\mathcal{H}_{L_{D_i}}^n$, where τ is some positive constant such that $|\tau - 1| \lesssim \alpha(D_i)$ and $\tau \lesssim 1$. Therefore,

$$\begin{aligned}
 (4.26) \quad & \left| \int g d(\mathcal{H}_{L_{D_i}}^n - \mathcal{H}_{L_{D_{i+1}}}^n) \right| = \left| \int (g(t) - \tau g(p(t))) d\mathcal{H}_{L_{D_i}}^n(t) \right| \\
 & \leq \left| \int (1 - \tau)g(t) d\mathcal{H}_{L_{D_i}}^n(t) \right| + \left| \int \tau(g(t) - g(p(t))) d\mathcal{H}_{L_{D_i}}^n(t) \right| \\
 & \lesssim 2^{-j(n+1)}\alpha(D_i) + \int |(g(t) - g(p(t)))| d\mathcal{H}_{L_{D_i}}^n(t).
 \end{aligned}$$

Since g and $g \circ p$ are supported in B_{D_j} and g is 1-Lipschitz, by [To, Lemma 3.4],

$$\begin{aligned}
 \int |(g - g \circ p)| d\mathcal{H}_{L_{D_i}}^n & \lesssim \int_{B_{D_j}} \text{dist}_{\mathcal{H}}(L_{D_i} \cap B_{D_j}, L_{D_{i+1}} \cap B_{D_j}) d\mathcal{H}_{L_{D_i}}^n \\
 & \lesssim 2^{-jn} \text{dist}_{\mathcal{H}}(L_{D_i} \cap B_{D_j}, L_{D_{i+1}} \cap B_{D_j}) \\
 & \lesssim 2^{-jn} 2^{i-j} \text{dist}_{\mathcal{H}}(L_{D_i} \cap B_{D_i}, L_{D_{i+1}} \cap B_{D_i}) \lesssim 2^{-j(n+1)}\alpha(D_i).
 \end{aligned}$$

This last estimate together with (4.26) and the fact that $|c_{D_{i+1}}| \lesssim 1$ implies that the second term on the right hand side of (4.25) is also bounded in absolute value by $C2^{-j(n+1)}\alpha(D_i)$. Therefore, to obtain (4.24) we only have to take the supremum in (4.25) over all admissible functions g .

4.1.3. Estimate of $F3_j^m(x)$. Notice that, by Fubini's theorem,

$$\begin{aligned}
 \Lambda_m^\mu(x, z; y) - \Lambda_m^{\sigma_D}(x, z; y) &= 2^{mn} \int_{\{a: x, z \in D_m^a, y \notin D_m^a\}} \left(\frac{1}{\mu(D_m^a)} - \frac{1}{\sigma_D(D_m^a)} \right) da \\
 &= 2^{mn} \int_{\{a: x, z \in D_m^a, y \notin D_m^a\}} \frac{\sigma_D(D_m^a) - \mu(D_m^a)}{\mu(D_m^a)\sigma_D(D_m^a)} da \\
 &= 2^{mn} \int_{\{a: x, z \in D_m^a, y \notin D_m^a\}} \left(\int_{t \in D_m^a} d(\sigma_D - \mu)(t) \right) \mu(D_m^a)^{-1} \sigma_D(D_m^a)^{-1} da \\
 &= \int \Lambda_m^{\mu, \sigma_D}(x, z, t; y) d(\sigma_D - \mu)(t).
 \end{aligned}$$

Since $\Lambda_m^{\mu, \sigma_D}(x, z, t; y) = 0$ if $|\tilde{x} - \tilde{t}| > 2^{-m}\sqrt{n}$, we may assume that $\text{supp} \Lambda_m^{\mu, \sigma_D}(x, z, \cdot; y) \subset B_D$ (by taking C_Γ big enough).

Claim 4.3. *We have $|\Lambda_m^{\mu, \sigma_D}(x, z, t; y)| \lesssim 2^{2mn}$ and $|\nabla_t \Lambda_m^{\mu, \sigma_D}(x, z, t; y)| \lesssim 2^{m(2n+1)}$ for all $x, y, z, t \in \mathbb{R}^d$.*

Using Claim 4.3, we deduce that $|\Lambda_m^\mu(x, z; y) - \Lambda_m^{\sigma_D}(x, z; y)| \lesssim 2^{m(2n+1)} \text{dist}_{B_D}(\mu, \sigma_D)$, and then,

$$\begin{aligned}
 |F3_j^m(x)| & \lesssim \int \varphi_{2^{-j-1}}^{2^{-j}}(x - y) \int_{|\tilde{x} - \tilde{z}| \leq 2^{-m}\sqrt{n}} 2^{m(2n+1)} \text{dist}_{B_D}(\mu, \sigma_D) \\
 & \quad |x - z| |x - y|^{-n-1} d\sigma_D(z) d\sigma_D(y) \\
 & \lesssim 2^{2mn+j(n+1)} \text{dist}_{B_D}(\mu, \sigma_D) \int \int_{\substack{|\tilde{x} - \tilde{y}| \leq 2^{-j}3\sqrt{n} \\ |\tilde{x} - \tilde{z}| \leq 2^{-m}\sqrt{n}}} d\sigma_D(z) d\sigma_D(y) \\
 & \lesssim 2^{mn+j} \text{dist}_{B_D}(\mu, \sigma_D) \lesssim 2^{j-m} \alpha(D),
 \end{aligned}$$

which, together with the estimate of $|F1_j^m(x)|$ (see subsection 4.1.1), gives (4.15).

4.1.4. Estimate of $F4_j^m(x)$. Set $L_D = \{(\tilde{y}, a(\tilde{y})) \in \mathbb{R}^d : \tilde{y} \in \mathbb{R}^n\}$, where $a : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$ is an appropriate affine map, and let $p : L_D^x \rightarrow L_D$ be the projection defined by $p(y) := (\tilde{y}, a(\tilde{y}))$. Since Γ is a Lipschitz graph, a is well defined and p is a homeomorphism. If $p_\# \sigma_D^x$ is the image measure of σ_D^x under p , we obviously have $\sigma_D = p_\# \sigma_D^x$ because L_D and L_D^x differ by a translation. Therefore, since $\widetilde{p(y)} = \tilde{y}$, (4.14) becomes

$$F4_j^m(x) = \iint (K(x - p(y)) - K(p(z) - p(y)) - (K(x - y) - K(z - y))) \varphi_{2^{-j-1}}^{2-j}(x - y) \Lambda_m^{\sigma_D^x}(x, z; y) d\sigma_D^x(z) d\sigma_D^x(y).$$

For $y, z \in L_D^x$ such that $\varphi_{2^{-j-1}}^{2-j}(x - y) \Lambda_m^{\sigma_D^x}(x, z; y) \neq 0$, we have $K(p(z) - p(y)) - K(z - y) = 0$, so we can estimate

$$\begin{aligned} |K(x - p(y)) - K(p(z) - p(y)) - (K(x - y) - K(z - y))| &= |K(x - p(y)) - K(x - y)| \\ &\lesssim \frac{|y - p(y)|}{|x - y|^{n+1}} \lesssim 2^{j(n+1)} |y - p(y)| \approx 2^{j(n+1)} \text{dist}(x, L_D). \end{aligned}$$

By the same arguments as in the proof of Claim 4.1, one can easily see that $|\Lambda_m^{\sigma_D^x}(x, z; y)| \lesssim 2^{mn}$. Therefore,

$$\begin{aligned} |F4_j^m(x)| &\lesssim 2^{j(n+1)} \text{dist}(x, L_D) 2^{mn} \iint_{\substack{|\tilde{x} - \tilde{y}| \leq 2^{-j} 3\sqrt{n} \\ |\tilde{x} - \tilde{z}| \leq 2^{-m} \sqrt{n}}} d\sigma_D^x(z) d\sigma_D^x(y) \\ &\lesssim 2^j \text{dist}(x, L_D) \approx 2^{j-m} \text{dist}(x, L_D) / \ell(D), \end{aligned}$$

which gives (4.17).

4.2. Estimate of $\sum_{j \in \mathbb{Z}} G_j^m(x)$ when $x \in D \cap \Gamma$ for some $D \in \mathcal{D}_m$. Assume that $x \in D$ for some $D \in \mathcal{D}_m$. Recall from (4.3) that

$$G_j^m(x) = \iint \varphi_{2^{-j-1}}^{2-j}(z - y) \gamma_{2^{-m}}(x - y) \Lambda_m^\mu(x, z; y) K(z - y) d\mu(z) d\mu(y),$$

where $0 \leq \gamma_{2^{-m}}(x - y) \leq 1$, $|\nabla_y \gamma_{2^{-m}}(x - y)| \lesssim 2^m$ for all $x, y \in \mathbb{R}^d$, and $\gamma_{2^{-m}}(x - y) = 0$ whenever $|\tilde{x} - \tilde{y}| > 2^{-m} 3\sqrt{n}$. Notice that $\Lambda_m^\mu(x, z; y) = 0$ if $|\tilde{x} - \tilde{z}| > 2^{-m} \sqrt{n}$, thus we can assume that $|\tilde{x} - \tilde{z}| \leq 2^{-m} \sqrt{n}$ and $|\tilde{z} - \tilde{y}| \leq 2^{-m+2} \sqrt{n}$ in the integral that defines $G_j^m(x)$. Hence, if $j \leq m - 2$, $\varphi_{2^{-j-1}}^{2-j}(z - y) \Lambda_m^\mu(x, z; y) = 0$ for all $z, y \in \mathbb{R}^d$, because $\varphi_{2^{-j-1}}^{2-j}(z - y) = 0$ if $|\tilde{z} - \tilde{y}| \leq 2^{-j-1} 2.1\sqrt{n}$, and $2^{-j-1} 2.1\sqrt{n} \geq 2^{-m+2} \sqrt{n}$ when $j \leq m - 2$. Therefore, $G_j^m(x) = 0$ for $j \leq m - 2$, and then

$$(4.27) \quad \sum_{j \in \mathbb{Z}} G_j^m(x) = \sum_{j \geq m-1} G_j^m(x);$$

so, from now on, we assume that $j \geq m - 1$.

Let L_D be an n -plane that minimizes $\alpha(D)$ and let $\sigma_D := c_D \mathcal{H}_{L_D}^n$ be a minimizing measure of $\alpha(D)$. As we did in the beginning of subsection 4.1, given $a \in \mathbb{R}^n$, let $b := a + \{2^{-m-1}\}^n \in \mathbb{R}^n$ be the center of \tilde{D}_m^a . Recall that, for $t \in \mathbb{R}^d$, $t \in \overline{D}_m^a$ if and only if $\|\tilde{t} - b\|_\infty \leq 2^{-m}$. Using that σ_D is a Hausdorff measure on an n -plane, that K is antisymmetric and that $\varphi_{2^{-j-1}}^{2-j}$ and

$\gamma_{2^{-m}}$ are symmetric, one can show that

$$0 = \int_{\|\tilde{x}-b\|_\infty \leq 2^{-m}} \int_{\|\tilde{z}-b\|_\infty \leq 2^{-m}} \int_{\|\tilde{y}-b\|_\infty > 2^{-m}} \varphi_{2^{-j-1}}^{2^{-j}}(z-y) \gamma_{2^{-m}}(x-y) K(z-y) d\sigma_D(y) d\sigma_D(z) db.$$

By the change of variable $b = a + \{2^{-m-1}\}^n$, it is easy to see that this triple integral is equal to $\int_{\{a: x \in D_m^a\}} \int_{D_m^a} \int_{(D_m^a)^c} \varphi_{2^{-j-1}}^{2^{-j}}(z-y) \gamma_{2^{-m}}(x-y) K(z-y) d\sigma_D(y) d\sigma_D(z) da$. Thus, since $\sigma_D(D_m^a)$ does not depend on $a \in \mathbb{R}^n$ because σ_D is flat,

$$\begin{aligned} 0 &= \int_{\{a: x \in D_m^a\}} \frac{2^{mn}}{\sigma_D(D_m^a)} \int_{D_m^a} \int_{(D_m^a)^c} \varphi_{2^{-j-1}}^{2^{-j}}(z-y) \gamma_{2^{-m}}(x-y) K(z-y) d\sigma_D(y) d\sigma_D(z) da \\ (4.28) \quad &= \iint \varphi_{2^{-j-1}}^{2^{-j}}(z-y) \gamma_{2^{-m}}(x-y) \Lambda_m^{\sigma_D}(x, z; y) K(z-y) d\sigma_D(z) d\sigma_D(y). \end{aligned}$$

Let $\{\eta_Q\}_{Q \in \mathcal{D}_j}$ be a partition of the unity with respect to the v-cubes $Q \in \mathcal{D}_j$, i.e. $\eta_Q : \mathbb{R}^d \rightarrow \mathbb{R}$ are \mathcal{C}^∞ functions such that: $\chi_{0.9Q} \leq \eta_Q \leq \chi_{1.1Q}$, $|\nabla \eta_Q| \lesssim \ell(Q)^{-1} = 2^j$, $\sum_{Q \in \mathcal{D}_j} \eta_Q = 1$ and $\eta_Q(y) = \eta_Q(\tilde{y}, 0)$ for all $y \in \mathbb{R}^d$. It is easy to check that, if $j \geq m-1$, $Q \in \mathcal{D}_j$, and $\text{supp} \eta_Q \cap \text{supp} \gamma_{2^{-m}}(x - \cdot) \neq \emptyset$, then $Q \subset C_e D$ for some absolute constant $C_e > 1$.

Given $Q \in \mathcal{D}_j$, let L_Q and $\sigma_Q := c_Q \mathcal{H}_{L_Q}^n$ be a minimizing n -plane and measure for $\alpha(Q)$, respectively, and consider the measure

$$\lambda := \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \eta_Q \sigma_Q.$$

By (4.28) and the properties of the partition of the unity $\{\eta_Q\}_{Q \in \mathcal{D}_j}$, for $j \geq m-1$ we can decompose $G_j^m(x)$ as

$$(4.29) \quad G_j^m(x) = G1_j^m(x) + G2_j^m(x) + G3_j^m(x) + G4_j^m(x) + G5_j^m(x),$$

where

$$(4.30) \quad G1_j^m(x) := \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \iint \dots d(\mu - \sigma_Q)(z) d\mu(y),$$

$$(4.31) \quad G2_j^m(x) := \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \iint \dots d\sigma_Q(z) d(\mu - \sigma_Q)(y),$$

$$(4.32) \quad G3_j^m(x) := \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \iint \dots d(\sigma_Q \times \sigma_Q - \sigma_D \times \sigma_D)(z, y),$$

where “...” stands for $\varphi_{2^{-j-1}}^{2^{-j}}(z-y) \gamma_{2^{-m}}(x-y) \eta_Q(y) K(z-y) \Lambda_m^\mu(x, z; y)$, and

$$(4.33) \quad G4_j^m(x) := \iint \varphi_{2^{-j-1}}^{2^{-j}}(z-y) \gamma_{2^{-m}}(x-y) K(z-y) (\Lambda_m^\mu(x, z; y) - \Lambda_m^\lambda(x, z; y)) d\sigma_D(z) d\sigma_D(y),$$

$$(4.34) \quad G5_j^m(x) := \iint \varphi_{2^{-j-1}}^{2^{-j}}(z-y) \gamma_{2^{-m}}(x-y) K(z-y) (\Lambda_m^\lambda(x, z; y) - \Lambda_m^{\sigma_D}(x, z; y)) d\sigma_D(z) d\sigma_D(y).$$

In the next subsections we will prove the following estimates:

$$(4.35) \quad |G1_j^m(x)| + |G2_j^m(x)| + |G4_j^m(x)| \lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} 2^{(m-j)(n+1)} \alpha(Q),$$

$$(4.36) \quad |G3_j^m(x)| + |G5_j^m(x)| \lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} 2^{(m-j)(n+1)} \left(\alpha(C_b D) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_b D} \alpha(R) \right),$$

where C_b is some absolute constant bigger than C_e . Then, using (4.27) and (4.29), we will finally obtain that, for all $D \in \mathcal{D}_m$ and $x \in D \cap \Gamma$,

$$(4.37) \quad \begin{aligned} \sum_{j \in \mathbb{Z}} |G_j^m(x)| &\lesssim \sum_{j \geq m-1} \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} 2^{(m-j)(n+1)} \left(\alpha(C_b D) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_b D} \alpha(R) \right) \\ &\lesssim \sum_{Q \in \mathcal{D} : Q \subset C_e D} \frac{\ell(Q)^{n+1}}{\ell(D)^{n+1}} \left(\alpha(C_b D) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_b D} \alpha(R) \right) \\ &\lesssim \alpha(C_b D) + \sum_{R \in \mathcal{D} : R \subset C_b D} \frac{\ell(R)^{n+1}}{\ell(D)^{n+1}} \alpha(R), \end{aligned}$$

which gives (4.5).

4.2.1. Estimate of $G1_j^m(x)$. If $\varphi_{2^{-j-1}}^{2^{-j}}(z-y) \neq 0$ then $2^{-j-1}2.1\sqrt{n} \leq |\tilde{z} - \tilde{y}| \leq 2^{-j}3\sqrt{n}$, so if we also have that $y \in \text{supp}\eta_Q$, then $z \in 8\sqrt{n}Q$ because $Q \in \mathcal{D}_j$. Therefore, we can assume that $\text{supp}\varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)\eta_Q(y) \subset B_Q$ if C_Γ is big enough.

Claim 4.4. *For $z \in \text{supp}\varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, the following hold: $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)-j}$, $|\nabla_z \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$, and $|\nabla_y \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$.*

We have that $|K(z-y)| \lesssim 2^{jn}$ and $|\nabla_z K(z-y)| \lesssim 2^{j(n+1)}$ for all $z \in \text{supp}\varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$. Using (4.19) and the first two estimates in Claim 4.4, we get

$$|\nabla_z(\varphi_{2^{-j-1}}^{2^{-j}}(z-y)K(z-y)\Lambda_m^\mu(x, z; y))| \lesssim 2^{m(n+1)+jn}.$$

Therefore, for $y \in \text{supp}\eta_Q$,

$$\begin{aligned} \left| \int \varphi_{2^{-j-1}}^{2^{-j}}(z-y)K(z-y)\Lambda_m^\mu(x, z; y) d(\mu - \sigma_Q)(z) \right| &\lesssim 2^{m(n+1)+jn} \text{dist}_{B_Q}(\mu, \sigma_Q) \\ &\lesssim 2^{m(n+1)-j} \alpha(Q), \end{aligned}$$

and then,

$$|G1_j^m(x)| \lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} \int_{\text{supp}\eta_Q} 2^{m(n+1)-j} \alpha(Q) d\mu(y) \lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} 2^{(m-j)(n+1)} \alpha(Q).$$

4.2.2. Estimate of $G2_j^m(x)$. It can be estimated using the arguments of subsection 4.2.1, but now we also have to take into account that $|\nabla_y \gamma_{2^{-m}}(x-y)| \lesssim 2^m \lesssim 2^j$, because we are assuming $j \geq m-1$, and we have to use the last estimate in Claim 4.4.

4.2.3. **Estimate of $G3_j^m(x)$.** Given $x \in D \cap \Gamma$ and $Q \in \mathcal{D}_j$, denote

$$H_Q(y, z) := \varphi_{2^{-j-1}}^{2^{-j}}(z - y) \gamma_{2^{-m}}(x - y) \eta_Q(y) K(z - y) \Lambda_m^\mu(x, z; y).$$

Then, (4.32) becomes

$$\begin{aligned} G3_j^m(x) &= \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \iint H_Q(y, z) d(\sigma_Q \times \sigma_Q - \sigma_D \times \sigma_D)(z, y) \\ &= \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \iint H_Q(y, z) d(c_Q^2 \mathcal{H}_{L_Q}^n \times \mathcal{H}_{L_Q}^n - c_D^2 \mathcal{H}_{L_D}^n \times \mathcal{H}_{L_D}^n)(z, y) \\ (4.38) \quad &= \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \iint H_Q(y, z) (c_Q^2 - c_D^2) d\mathcal{H}_{L_Q}^n(z) d\mathcal{H}_{L_Q}^n(y) \\ &\quad + \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} c_D^2 \iint H_Q(y, z) d(\mathcal{H}_{L_Q}^n \times \mathcal{H}_{L_Q}^n - \mathcal{H}_{L_D}^n \times \mathcal{H}_{L_D}^n)(z, y) \\ &=: G3A_j^m(x) + G3B_j^m(x). \end{aligned}$$

We are going to estimate the terms $G3A_j^m(x)$ and $G3B_j^m(x)$ separately. Recall that $\ell(D) = 2^{-m}$. Given a v-cube $Q \in \mathcal{D}_j$ such that $Q \subset C_e D$, let $Q =: Q_j \subset \dots \subset Q_{i+1} \subset Q_i \subset \dots \subset Q_{m-1}$ be the sequence of v-cubes such that Q_i belongs to \mathcal{D}_i for $i = m-1, \dots, j$. Evidently, $Q_{m-1} \subset C_b D$ for some constant C_b big enough, because $\ell(Q_{m-1}) = 2\ell(D)$ and $Q \subset Q_{m-1} \cap C_e D$. Let L_{Q_i} be an n -plane that minimizes $\alpha(Q_i)$ and let $\sigma_{Q_i} := c_{Q_i} \mathcal{H}_{L_{Q_i}}^n$ be a minimizing measure of $\alpha(Q_i)$. Also, let $L_{C_b D}$ and $\sigma_{C_b D} := c_{C_b D} \mathcal{H}_{L_{C_b D}}^n$ be a minimizing n -plane and measure of $\alpha(C_b D)$, respectively.

In order to estimate $G3A_j^m(x)$, notice that, by [To, Lemma 3.4] and the triangle inequality, $|c_{Q_i}| \lesssim 1$ for all $i = m-1, \dots, j$, and

$$\begin{aligned} |c_Q^2 - c_D^2| &= |c_Q + c_D| |c_Q - c_D| \lesssim |c_{Q_j} - c_D| \\ (4.39) \quad &\lesssim |c_{Q_{m-1}} - c_{C_b D}| + |c_{C_b D} - c_D| + \sum_{i=m-1}^{j-1} |c_{Q_{i+1}} - c_{Q_i}| \\ &\lesssim \alpha(C_b D) + \sum_{i=m-1}^{j-1} \alpha(Q_i) \lesssim \alpha(C_b D) + \sum_{R \in \mathcal{D}: Q \subset R \subset C_b D} \alpha(R) \end{aligned}$$

(in the case that $j = m-1$, there are no intermediate scales between j and $m-1$, so one just obtains $|c_Q^2 - c_D^2| \lesssim \alpha(C_b D)$).

Claim 4.5. *For $z \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, we have $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)-j}$.*

Notice that this last estimate is the same as the first one in Claim 4.4. Using Claim 4.5 and that $|K(z - y)| \lesssim 2^{jn}$ for all $z \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, we easily obtain $|H_Q(y, z)| \lesssim 2^{m(n+1)+j(n-1)}$. Therefore, using (4.39),

$$\begin{aligned} |G3A_j^m(x)| &\lesssim \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} |c_Q^2 - c_D^2| \iint |H_Q(y, z)| d\mathcal{H}_{L_Q}^n(z) d\mathcal{H}_{L_Q}^n(y) \\ (4.40) \quad &\lesssim \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} 2^{(m-j)(n+1)} \left(\alpha(C_b D) + \sum_{R \in \mathcal{D}: Q \subset R \subset C_b D} \alpha(R) \right). \end{aligned}$$

To estimate $G3B_j^m(x)$ in (4.38), we set

$$(4.41) \quad G3B_j^m(x) = \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} c_D^2 G3B(Q)_j^m(x),$$

where $G3B(Q)_j^m(x) := \iint H_Q d(\mathcal{H}_{L_Q}^n \times \mathcal{H}_{L_Q}^n - \mathcal{H}_{L_D}^n \times \mathcal{H}_{L_D}^n)$. Given $Q \in \mathcal{D}_j$ such that $Q \subset C_e D$, we split $G3B(Q)_j^m(x)$ as follows:

$$(4.42) \quad \begin{aligned} G3B(Q)_j^m(x) &= \sum_{i=m-1}^{j-1} \iint H_Q d(\mathcal{H}_{L_{Q_{i+1}}}^n \times \mathcal{H}_{L_{Q_{i+1}}}^n - \mathcal{H}_{L_{Q_i}}^n \times \mathcal{H}_{L_{Q_i}}^n) \\ &\quad + \iint H_Q d(\mathcal{H}_{L_{Q_{m-1}}}^n \times \mathcal{H}_{L_{Q_{m-1}}}^n - \mathcal{H}_{L_{C_b D}}^n \times \mathcal{H}_{L_{C_b D}}^n) \\ &\quad + \iint H_Q d(\mathcal{H}_{L_{C_b D}}^n \times \mathcal{H}_{L_{C_b D}}^n - \mathcal{H}_{L_D}^n \times \mathcal{H}_{L_D}^n) \end{aligned}$$

(as before, in the case $j = m - 1$ the first term on the right hand side of (4.42) does not exist).

Fix $i \in \mathbb{Z}$ such that $m - 1 \leq i < j$. Set $L_{Q_{i+1}} = \{(\tilde{y}, a(\tilde{y})) \in \mathbb{R}^d : \tilde{y} \in \mathbb{R}^n\}$, where $a : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$ is an appropriate affine map, and let $p : L_{Q_i} \rightarrow L_{Q_{i+1}}$ be the map defined by $p(y) := (\tilde{y}, a(\tilde{y}))$. Let $p_{\#} \mathcal{H}_{L_{Q_i}}^n$ be the image measure of $\mathcal{H}_{L_{Q_i}}^n$ by p . It is easy to check that $\mathcal{H}_{L_{Q_{i+1}}}^n = \tau p_{\#} \mathcal{H}_{L_{Q_i}}^n$, where τ is some positive constant such that $|\tau - 1| \lesssim \alpha(Q_i)$ and $\tau \lesssim 1$. Therefore,

$$(4.43) \quad \begin{aligned} &\iint H_Q(y, z) d(\mathcal{H}_{L_{Q_{i+1}}}^n \times \mathcal{H}_{L_{Q_{i+1}}}^n - \mathcal{H}_{L_{Q_i}}^n \times \mathcal{H}_{L_{Q_i}}^n)(z, y) \\ &= \iint \left(\tau^2 H_Q(p(y), p(z)) - H_Q(y, z) \right) d\mathcal{H}_{L_{Q_i}}^n(z) d\mathcal{H}_{L_{Q_i}}^n(y) \\ &= \iint \tau^2 \left(H_Q(p(y), p(z)) - H_Q(y, z) \right) d\mathcal{H}_{L_{Q_i}}^n(z) d\mathcal{H}_{L_{Q_i}}^n(y) \\ &\quad + \iint (\tau^2 - 1) H_Q(y, z) d\mathcal{H}_{L_{Q_i}}^n(z) d\mathcal{H}_{L_{Q_i}}^n(y). \end{aligned}$$

Since $|\tau^2 - 1| \lesssim \alpha(Q_j)$ and we have seen that $|H_Q(y, z)| \lesssim 2^{m(n+1)+j(n-1)}$ after Claim 4.5, the second term on the right side of the last equality is bounded by $C2^{(m-j)(n+1)}\alpha(Q_i)$.

In order to estimate the first term on the right hand side of (4.43), notice that $\varphi_{2^{-j-1}}^{2-j}(z - y)$, $\gamma_{2^{-m}}(x - y)$, $\eta_Q(y)$ and $\Lambda_m^\mu(x, z; y)$ only depend on the first n coordinates of y and z (i.e., on \tilde{y} and \tilde{z}), thus their values coincide on (y, z) and $(p(y), p(z))$. Then,

$$\begin{aligned} &\iint \tau^2 \left(H_Q(p(y), p(z)) - H_Q(y, z) \right) d\mathcal{H}_{L_{Q_i}}^n(z) d\mathcal{H}_{L_{Q_i}}^n(y) \\ &= \tau^2 \iint \varphi_{2^{-j-1}}^{2-j}(z - y) \gamma_{2^{-m}}(x - y) \eta_Q(y) \Lambda_m^\mu(x, z; y) \\ &\quad \left(K(p(z) - p(y)) - K(z - y) \right) d\mathcal{H}_{L_{Q_i}}^n(z) d\mathcal{H}_{L_{Q_i}}^n(y). \end{aligned}$$

Let θ_i be the angle between L_{Q_i} and $L_{Q_{i+1}}$. One can easily see that, for $y, z \in L_{Q_i}$, $|(p(z) - p(y)) - (z - y)| \lesssim \sin(\theta_i)|z - y| \lesssim \alpha(Q_i)|z - y|$. Thus, if also $z \in \text{supp} \varphi_{2^{-j-1}}^{2-j}(\cdot - y)$,

$$|K(p(z) - p(y)) - K(z - y)| \lesssim 2^{j(n+1)}|(p(z) - p(y)) - (z - y)| \lesssim 2^{jn}\alpha(Q_i).$$

Together with Claim 4.5 and the fact that $\tau^2 \lesssim 1$, this gives

$$\begin{aligned} & \left| \iint \tau^2 \left(H_Q(p(y), p(z)) - H_Q(y, z) \right) d\mathcal{H}_{L_{Q_i}}^n(z) d\mathcal{H}_{L_{Q_i}}^n(y) \right| \\ & \lesssim \iint \varphi_{2^{-j-1}}^{2-j}(z-y) \eta_Q(y) 2^{m(n+1)+j(n-1)} \alpha(Q_i) d\mathcal{H}_{L_{Q_i}}^n(z) d\mathcal{H}_{L_{Q_i}}^n(y) \\ & \lesssim 2^{(m-j)(n+1)} \alpha(Q_i). \end{aligned}$$

From the last estimates and (4.43), we get

$$\left| \iint H_Q d(\mathcal{H}_{L_{Q_{i+1}}}^n \times \mathcal{H}_{L_{Q_{i+1}}}^n - \mathcal{H}_{L_{Q_i}}^n \times \mathcal{H}_{L_{Q_i}}^n) \right| \lesssim 2^{(m-j)(n+1)} \alpha(Q_i)$$

for $i = m-1, \dots, j-1$. With similar arguments, one also obtains

$$\begin{aligned} & \left| \iint H_Q d(\mathcal{H}_{L_{Q_{m-1}}}^n \times \mathcal{H}_{L_{Q_{m-1}}}^n - \mathcal{H}_{L_{C_b D}}^n \times \mathcal{H}_{L_{C_b D}}^n) \right| \lesssim 2^{(m-j)(n+1)} \alpha(C_b D), \\ & \left| \iint H_Q d(\mathcal{H}_{L_{C_b D}}^n \times \mathcal{H}_{L_{C_b D}}^n - \mathcal{H}_{L_D}^n \times \mathcal{H}_{L_D}^n) \right| \lesssim 2^{(m-j)(n+1)} \alpha(C_b D). \end{aligned}$$

These last three inequalities together with (4.42), (4.41) and the fact that $|c_D| \lesssim 1$ yield

$$\begin{aligned} (4.44) \quad |G3B_j^m(x)| & \lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} 2^{(m-j)(n+1)} \left(\alpha(C_b D) + \sum_{i=m-1}^{j-1} \alpha(Q_i) \right) \\ & \leq \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} 2^{(m-j)(n+1)} \left(\alpha(C_b D) + \sum_{R \in \mathcal{D} : Q \subset R \subset C_b D} \alpha(R) \right). \end{aligned}$$

Finally, (4.40) and (4.44) applied to (4.38) give half of (4.36).

4.2.4. Estimate of $G4_j^m(x)$. By Fubini's theorem and the definitions of λ , Λ_m^μ and Λ_m^λ ,

$$\begin{aligned} \Lambda_m^\mu(x, z; y) - \Lambda_m^\lambda(x, z; y) & = 2^{mn} \int_{\{a: x, z \in D_m^a, y \notin D_m^a\}} \frac{\lambda(D_m^a) - \mu(D_m^a)}{\mu(D_m^a) \lambda(D_m^a)} da \\ & = 2^{mn} \int_{\{a: x, z \in D_m^a, y \notin D_m^a\}} \left(\int_{t \in D_m^a} \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} \eta_Q(t) d(\sigma_Q - \mu)(t) \right) \frac{da}{\mu(D_m^a) \lambda(D_m^a)} \\ & = \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} \int \eta_Q(t) \Lambda_m^{\mu, \lambda}(x, z, t; y) d(\sigma_Q - \mu)(t). \end{aligned}$$

We also used in the second equality that $1 = \sum_{Q \in \mathcal{D}_j} \eta_Q(t) = \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} \eta_Q(t)$ for all $t \in D_m^a$ if C_e is big enough, and this is because $j \geq m-1$ and $|\tilde{x} - \tilde{t}| \lesssim 2^{-m}$ for all $t \in D_m^a$.

Claim 4.6. For $x \in D$, $j \geq m-1$, $|x-y| \lesssim 2^{-m}$, and $z \in \text{supp} \varphi_{2^{-j-1}}^{2-j}(\cdot - y)$, the following hold: $|\Lambda_m^{\mu, \lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)-j}$ and $|\nabla_t \Lambda_m^{\mu, \lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)}$.

Using Claim 4.6 and the properties of η_Q , we obtain

$$\begin{aligned} |\Lambda_m^\mu(x, z; y) - \Lambda_m^\lambda(x, z; y)| & \lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} 2^{m(2n+1)} \text{dist}_{B_Q}(\mu, \sigma_Q) \\ & \lesssim \sum_{Q \in \mathcal{D}_j : Q \subset C_e D} 2^{m(2n+1)-j(n+1)} \alpha(Q). \end{aligned}$$

Plugging this estimate into the definition of $G4_j^m(x)$ in (4.33), we get

$$\begin{aligned} |G4_j^m(x)| &\lesssim \iint \varphi_{2^{-j-1}}^{2^{-j}}(z-y) \gamma_{2^{-m}}(x-y) |K(z-y)| \\ &\quad \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} 2^{m(2n+1)-j(n+1)} \alpha(Q) d\sigma_D(z) d\sigma_D(y) \\ &\lesssim \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} 2^{(m-j)(n+1)} \alpha(Q), \end{aligned}$$

which, together with the estimates of $|G1_j^m(x)|$ and $|G2_j^m(x)|$ in subsections 4.2.1 and 4.2.2, gives (4.35).

4.2.5. Estimate of $G5_j^m(x)$. Arguing as in subsection 4.2.4, we have

$$\begin{aligned} \Lambda_m^\lambda(x, z; y) - \Lambda_m^{\sigma_D}(x, z; y) &= \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \int \eta_Q(t) \Lambda_m^{\lambda, \sigma_D}(x, z, t; y) d(\sigma_D - \sigma_Q)(t) \\ (4.45) \quad &= \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \int H_Q(t) d(\sigma_D - \sigma_Q)(t), \end{aligned}$$

where we have set $H_Q(t) := \eta_Q(t) \Lambda_m^{\lambda, \sigma_D}(x, z, t; y)$.

We are going to estimate the right hand side of (4.45) using the techniques of subsection 4.2.3. We have

$$(4.46) \quad \int H_Q d(\sigma_D - \sigma_Q) = (c_D - c_Q) \int H_Q d\mathcal{H}_{L_D}^n + c_Q \int H_Q d(\mathcal{H}_{L_D}^n - \mathcal{H}_{L_Q}^n).$$

We introduce the intermediate v-cubes between $Q \in \mathcal{D}_j$ and $D \in \mathcal{D}_m$ to obtain

$$(4.47) \quad |c_D - c_Q| \lesssim \alpha(C_b D) + \sum_{R \in \mathcal{D}: Q \subset R \subset C_b D} \alpha(R).$$

Claim 4.7. For $x \in D$, $j \geq m-1$, $|x-y| \lesssim 2^{-m}$, and $z \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, the following holds: $|\Lambda_m^{\lambda, \sigma_D}(x, z, t; y)| \lesssim 2^{m(2n+1)-j}$.

Combining Claim 4.7 with (4.47), we derive that

$$(4.48) \quad |c_D - c_Q| \int |H_Q| d\mathcal{H}_{L_D}^n \lesssim 2^{m(2n+1)-j(n+1)} \left(\alpha(C_b D) + \sum_{R \in \mathcal{D}: Q \subset R \subset C_b D} \alpha(R) \right).$$

For the second term on the right side of (4.46), one can also use the arguments in subsection 4.2.3 (see (4.42) and following) to show that

$$(4.49) \quad \left| \int H_Q d(\mathcal{H}_{L_D}^n - \mathcal{H}_{L_Q}^n) \right| \lesssim 2^{m(2n+1)-j(n+1)} \left(\alpha(C_b D) + \sum_{R \in \mathcal{D}: Q \subset R \subset C_b D} \alpha(R) \right)$$

(now it is easier because the function $H_Q(t)$ only depends on the first n coordinates of the points involved, i.e., it depends only on $\tilde{x}, \tilde{y}, \tilde{z}$ and \tilde{t} , so when we project vertically to deal with the image measure, the function H_Q is not affected). Therefore, by (4.48), (4.49), (4.46), and (4.45), we obtain

$$|\Lambda_m^\lambda(x, z; y) - \Lambda_m^{\sigma_D}(x, z; y)| \lesssim \sum_{\substack{Q \in \mathcal{D}_j: \\ Q \subset C_e D}} 2^{m(2n+1)-j(n+1)} \left(\alpha(C_b D) + \sum_{\substack{R \in \mathcal{D}: \\ Q \subset R \subset C_b D}} \alpha(R) \right).$$

From the definition of $G5_j^m(x)$ in (4.34), we conclude that

$$\begin{aligned} |G5_j^m(x)| &\lesssim \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} 2^{m(2n+1)-j(n+1)} \left(\alpha(C_b D) + \sum_{R \in \mathcal{D}: Q \subset R \subset C_b D} \alpha(R) \right) \\ &\quad \iint \varphi_{2^{-j-1}}^{2^{-j}}(z-y) \gamma_{2^{-m}}(x-y) |K(z-y)| d\sigma_D(z) d\sigma_D(y) \\ &\lesssim \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} 2^{(m-j)(n+1)} \left(\alpha(C_b D) + \sum_{R \in \mathcal{D}: Q \subset R \subset C_b D} \alpha(R) \right), \end{aligned}$$

which, together with the estimate of $|G3_j^m(x)|$ in subsection 4.2.3, gives (4.36).

4.3. Proof of Claims 4.1, ..., 4.7. We have to prove:

- **Claim 4.1:** We have $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{mn}$ and $|\nabla_z \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$ for all $x, y, z \in \mathbb{R}^d$.
- **Claim 4.2:** For $j < m$, $y \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(x - \cdot)$, and $|\tilde{x} - \tilde{z}| \leq 2^{-m} \sqrt{n}$, the following hold: $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{mn}$ and $\nabla_y \Lambda_m^\mu(x, z; y) = 0$.
- **Claim 4.3:** We have $|\Lambda_m^{\mu, \sigma D}(x, z, t; y)| \lesssim 2^{2mn}$ and $|\nabla_t \Lambda_m^{\mu, \sigma D}(x, z, t; y)| \lesssim 2^{m(2n+1)}$ for all $x, y, z, t \in \mathbb{R}^d$.
- **Claim 4.4:** For $z \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, the following hold: $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)-j}$, $|\nabla_z \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$, and $|\nabla_y \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$.
- **Claim 4.5:** For $z \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)-j}$.
- **Claim 4.6:** For $x \in D$, $j \geq m-1$, $|x-y| \lesssim 2^{-m}$, and $z \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, the following hold: $|\Lambda_m^{\mu, \lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)-j}$ and $|\nabla_t \Lambda_m^{\mu, \lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)}$.
- **Claim 4.7:** For $x \in D$, $j \geq m-1$, $|x-y| \lesssim 2^{-m}$, and $z \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, the following holds: $|\Lambda_m^{\lambda, \sigma D}(x, z, t; y)| \lesssim 2^{m(2n+1)-j}$.

To prove the claims, we need to express the function Λ at the end of subsection 2.3 in a more convenient way. Notice that we can replace D_m^a by $\overline{D_m^a}$ in the definition of Λ because μ and the n -dimensional Hausdorff measure vanish on ∂D_m^a .

For $u \in \mathbb{R}^n$ and $r > 0$, we denote $|u|_\infty := \max_{i=1, \dots, n} |u^i|$, $B_\infty(u, r) := \{v \in \mathbb{R}^n : |u-v|_\infty \leq r\}$, and $B_\infty^m(u) := B_\infty(u, 2^{-m-1})$. Given $a \in \mathbb{R}^n$, let $b := a + \{2^{-m-1}\}^n \in \mathbb{R}^n$ be the center of \tilde{D}_m^a . Then, given $q \in \mathbb{R}^d$,

$$q \in \overline{D_m^a} \iff |\tilde{q} - b|_\infty \leq 2^{-m} \iff b \in B_\infty^m(\tilde{q}).$$

Let μ_1, \dots, μ_k be positive Borel measures such that $\mu_l(D_m^a) > 0$ and $\mu_l(\partial D_m^a) = 0$ for all $a \in \mathbb{R}^n$, $m \in \mathbb{Z}$ and $l = 1, \dots, k$. Given $m \in \mathbb{Z}$ and $x_1, \dots, x_i, y_1, \dots, y_j \in \mathbb{R}^d$ we have, by the change of variable $b = a + \{2^{-m-1}\}^n \in \mathbb{R}^n$,

$$\begin{aligned} \Lambda_m^{\mu_1, \dots, \mu_k}(x_1, \dots, x_i; y_1, \dots, y_j) &= \int_{\{a \in \mathbb{R}^n : x_1, \dots, x_i \in \overline{D_m^a}, y_1, \dots, y_j \notin \overline{D_m^a}\}} \frac{2^{nm} da}{\prod_{l=1}^k \mu_l(D_m^a)} \\ (4.50) \quad &= 2^{nm} \int \frac{\chi_{B_\infty^m(\tilde{x}_1) \cap \dots \cap B_\infty^m(\tilde{x}_i) \cap B_\infty^m(\tilde{y}_1)^c \cap \dots \cap B_\infty^m(\tilde{y}_j)^c}(b)}{\prod_{l=1}^k \mu_l(D_m^{b - \{2^{-m-1}\}^n})} db. \end{aligned}$$

Proof of Claim 4.1. By (4.50), we have

$$(4.51) \quad \Lambda_m^\mu(x, z; y) = 2^{nm} \int \mu(D_m^{b - \{2^{-m-1}\}^n})^{-1} \chi_{B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{y})^c}(b) db.$$

Since $\mu(D_m^b) \gtrsim 2^{-mn}$ for all $b \in \mathbb{R}^n$,

$$|\Lambda_m^\mu(x, z; y)| \lesssim 2^{2mn} \mathcal{L}^n(B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{y})^c) \leq 2^{2mn} \mathcal{L}^n(B_\infty^m(\tilde{x})) \leq 2^{mn}.$$

To deal with the second inequality in Claim 4.1, we will estimate

$$|\Lambda_m^\mu(x, z_1; y) - \Lambda_m^\mu(x, z_2; y)| / |z_1 - z_2|$$

for z_1 close enough to z_2 . Recall that, given two sets $F_1, F_2 \subset \mathbb{R}^n$, $F_1 \Delta F_2 := (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ denotes their symmetric difference. Using (4.51), we get

$$\begin{aligned} & |\Lambda_m^\mu(x, z_1; y) - \Lambda_m^\mu(x, z_2; y)| \\ (4.52) \quad & \lesssim 2^{2nm} \int |\chi_{B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}_1) \cap B_\infty^m(\tilde{y})^c}(b) - \chi_{B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}_2) \cap B_\infty^m(\tilde{y})^c}(b)| db \\ & = 2^{2nm} \mathcal{L}^n\left((B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}_1) \cap B_\infty^m(\tilde{y})^c) \Delta (B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}_2) \cap B_\infty^m(\tilde{y})^c)\right) \\ & \leq 2^{2nm} \mathcal{L}^n(B_\infty^m(\tilde{z}_1) \Delta B_\infty^m(\tilde{z}_2)) \lesssim 2^{2nm} |\tilde{z}_1 - \tilde{z}_2| 2^{-m(n-1)} \leq 2^{m(n+1)} |z_1 - z_2|, \end{aligned}$$

and the claim follows. \square

Proof of Claim 4.2. The first estimate has been already proved in Claim 4.1. Let us deal with the second one. Notice that if $y \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(x - \cdot)$ then $|\tilde{x} - \tilde{y}| \geq 2^{-j-1} 2.1 \sqrt{n}$. Thus, if also $j < m$ and $|\tilde{x} - \tilde{z}| \leq 2^{-m} \sqrt{n}$, then $|\tilde{x} - \tilde{y}| > 2^{-m} \sqrt{n}$ and $|\tilde{z} - \tilde{y}| > 2^{-m} \sqrt{n}$. Therefore, $B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{y})^c = B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z})$ for all $y \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(x - \cdot)$, if $|\tilde{x} - \tilde{z}| \leq 2^{-m} \sqrt{n}$. This means, using (4.51), that $\Lambda_m^\mu(x, z; y)$ does not depend on y , so $\nabla_y \Lambda_m^\mu(x, z; y) = 0$ for all $y \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(x - \cdot)$, and the claim is proved. \square

Proof of Claim 4.3. This claim follows by arguments very similar to the ones in the proof of Claim 4.1. Just notice that $\mu(D_m^b) \sigma_D(D_m^b) \gtrsim 2^{-2mn}$ for all $b \in \mathbb{R}^n$. \square

Proof of Claim 4.4. Using (4.51), we have that

$$|\Lambda_m^\mu(x, z; y)| \lesssim 2^{2mn} \mathcal{L}^n(B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{y})^c) \leq 2^{2mn} \mathcal{L}^n(B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{y})^c).$$

Notice that $\mathcal{L}^n(B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{y})^c) \lesssim 2^{-m(n-1)} |\tilde{y} - \tilde{z}|$. Since $z \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$, $|\tilde{y} - \tilde{z}| \leq 2^{-j} 3 \sqrt{n}$. Thus, $\mathcal{L}^n(B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{y})^c) \lesssim 2^{-m(n-1)-j}$, and then $|\Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)-j}$.

In Claim 4.1 we already proved that $|\nabla_z \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$. Finally, to prove that $|\nabla_y \Lambda_m^\mu(x, z; y)| \lesssim 2^{m(n+1)}$, one can repeat the computations done in (4.52) but applied to the y coordinate and use that $B_\infty^m(\tilde{y}_1)^c \Delta B_\infty^m(\tilde{y}_2)^c = B_\infty^m(\tilde{y}_1) \Delta B_\infty^m(\tilde{y}_2)$. \square

Proof of Claim 4.5. This claim is included in the previous one. \square

Proof of Claim 4.6. Recall that $\lambda = \sum_{Q \in \mathcal{D}_j: Q \subset C_e D} \eta_Q \sigma_Q$, where C_e is some fixed constant big enough (see the beginning of subsection 4.2). Using the properties of η_Q and that C_e is big enough, it is not difficult to show that $\lambda(D_m^{b-\{2^{-m-1}\}^n}) \gtrsim 2^{-mn}$ for all $b \in \mathbb{R}^n$ such that $b \in B_\infty^m(\tilde{x})$ (recall that $x \in D$ and $j \geq m-1$). Therefore, by (4.50),

$$\begin{aligned} |\Lambda_m^{\mu, \lambda}(x, z, t; y)| & \lesssim 2^{3nm} \mathcal{L}^n(B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{t}) \cap B_\infty^m(\tilde{y})^c) \\ & \leq 2^{3nm} \mathcal{L}^n(B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{y})^c). \end{aligned}$$

As in the proof of Claim 4.4, we have $\mathcal{L}^n(B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{y})^c) \lesssim 2^{-m(n-1)-j}$ for all $z \in \text{supp} \varphi_{2^{-j-1}}^{2^{-j}}(\cdot - y)$. Thus, $|\Lambda_m^{\mu, \lambda}(x, z, t; y)| \lesssim 2^{m(2n+1)-j}$, as wished.

For the second estimate in Claim 4.6, we argue as in (4.52). For t_1 and t_2 close enough,

$$\begin{aligned} & |\Lambda_m^{\mu,\lambda}(x, z, t_1; y) - \Lambda_m^{\mu,\lambda}(x, z, t_2; y)| \\ & \lesssim 2^{3nm} \int |\chi_{B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{t}_1) \cap B_\infty^m(\tilde{y})^c}(b) - \chi_{B_\infty^m(\tilde{x}) \cap B_\infty^m(\tilde{z}) \cap B_\infty^m(\tilde{t}_2) \cap B_\infty^m(\tilde{y})^c}(b)| db \\ & \leq 2^{3nm} \mathcal{L}^n(B_\infty^m(\tilde{t}_1) \Delta B_\infty^m(\tilde{t}_2)) \lesssim 2^{3nm} |\tilde{t}_1 - \tilde{t}_2| 2^{-m(n-1)} \leq 2^{m(2n+1)} |t_1 - t_2|, \end{aligned}$$

and the claim follows by letting $t_1 \rightarrow t_2$. \square

Proof of Claim 4.7. This claim is proved as the first estimate in Claim 4.6, replacing μ by σ_D (we only used that $\mu(D_m^b) \gtrsim 2^{-mn}$ for all $b \in \mathbb{R}^n$, which also holds for σ_D). \square

5. PROOF OF THEOREM 3.2

Given $x \in \Gamma$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers such that

$$(5.1) \quad S\mu(x)^2 \leq 2 \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}: \epsilon_m, \epsilon_{m+1} \in I_j} |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^2,$$

so $\{\epsilon_m\}_{m \in \mathbb{Z}}$ depends on x .

Fix $j \in \mathbb{Z}$ and assume that $x \in D$, for some $D \in \mathcal{D}_j$. Let L_D be an n -plane that minimizes $\alpha(D)$ and let $\sigma_D := c_D \mathcal{H}_{L_D}^n$ be a minimizing measure for $\alpha(D)$. Let L_D^x be the n -plane parallel to L_D which contains x , and set $\sigma_D^x := c_D \mathcal{H}_{L_D^x}^n$.

By the antisymmetry of the function $\varphi_{\epsilon_{m+1}}^{\epsilon_m} K$, and since σ_D^x is a Hausdorff measure on the n -plane L_D^x and $x \in L_D^x$, we have

$$(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \sigma_D^x)(x) = \int \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y) K(x - y) d\sigma_D^x(y) = 0$$

for all $m \in \mathbb{Z}$. Therefore, we can decompose

$$(5.2) \quad (K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x) = (K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (\mu - \sigma_D))(x) + (K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (\sigma_D - \sigma_D^x))(x).$$

For every $m \in \mathbb{Z}$ such that $\epsilon_m, \epsilon_{m+1} \in I_j$ we will prove the following inequalities:

$$(5.3) \quad |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (\mu - \sigma_D))(x)| \lesssim 2^j |\epsilon_m - \epsilon_{m+1}| \alpha(D),$$

$$(5.4) \quad |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (\sigma_D - \sigma_D^x))(x)| \lesssim 2^{2j} |\epsilon_m - \epsilon_{m+1}| \text{dist}(x, L_D).$$

Assume for a moment that these estimates hold. Then, by (5.2),

$$|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)| \lesssim 2^j |\epsilon_m - \epsilon_{m+1}| (\alpha(D) + 2^j \text{dist}(x, L_D)).$$

Then, using (5.1), we conclude that

$$\begin{aligned} \|S\mu\|_{L^2(\mu)}^2 & \leq 2 \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \sum_{m \in \mathbb{Z}: \epsilon_m, \epsilon_{m+1} \in I_j} |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * \mu)(x)|^2 d\mu(x) \\ & \lesssim \sum_{j \in \mathbb{Z}} \sum_{D \in \mathcal{D}_j} \int_D \left(\alpha(D) + \frac{\text{dist}(x, L_D)}{2^{-j}} \right)^2 \sum_{\substack{m \in \mathbb{Z}: \\ \epsilon_m, \epsilon_{m+1} \in I_j}} \left(\frac{|\epsilon_m - \epsilon_{m+1}|}{2^{-j}} \right)^2 d\mu(x) \\ & \lesssim \sum_{D \in \mathcal{D}} \alpha(D)^2 \mu(D) + \sum_{D \in \mathcal{D}} \int_D \left(\frac{\text{dist}(x, L_D)}{\ell(D)} \right)^2 d\mu(x). \end{aligned}$$

Notice that, under the integral sign on the right hand side of the first inequality above, ϵ_m and ϵ_{m+1} depend on x . It is not obvious that the resulting function is measurable. To deal with this issue more carefully, we might first ask $\{\epsilon_m\}_{m \in \mathbb{Z}}$ to lie in some finite set, prove

the variational norm inequality with constants independent of the set, and then enlarge the set. Then, by monotone convergence we would obtain the result with $\{\epsilon_m\}_{m \in \mathbb{Z}}$ restricted to a countable set dense in $(0, \infty)$. The final theorem would follow then from the continuity properties of the operators involved. This applies to other similar situations in the paper. However, for the sake of conciseness, we will skip further details.

The second term on the right hand side of the last inequality coincides with $W_1\mu$ (see (4.6)), thus it is bounded (modulo constants) by $\sum_{D \in \mathcal{D}} (\alpha(D)^2 + \beta_2(D)^2)\mu(D)$, and Theorem 3.2 is proved.

It only remains to verify (5.3) and (5.4) for $x \in D \in \mathcal{D}_j$ and $m \in \mathbb{Z}$ such that $\epsilon_m, \epsilon_{m+1} \in I_j$. First of all, notice that $\varphi_{\epsilon_{m+1}}^{\epsilon_m}$ satisfies

$$(5.5) \quad \begin{aligned} |\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y)| &= \left| \varphi_{\mathbb{R}}\left(\frac{|\tilde{x} - \tilde{y}|}{\epsilon_{m+1}}\right) - \varphi_{\mathbb{R}}\left(\frac{|\tilde{x} - \tilde{y}|}{\epsilon_m}\right) \right| \leq \|\varphi'_{\mathbb{R}}\|_{L^\infty(\mathbb{R})} \left| \frac{|\tilde{x} - \tilde{y}|}{\epsilon_{m+1}} - \frac{|\tilde{x} - \tilde{y}|}{\epsilon_m} \right| \\ &= \|\varphi'_{\mathbb{R}}\|_{\infty} |\tilde{x} - \tilde{y}| \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m \epsilon_{m+1}} \lesssim 2^j |\epsilon_m - \epsilon_{m+1}| \end{aligned}$$

for all $y \in \text{supp } \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot)$. For $i = 1, \dots, d$,

$$\partial_{y^i}(\varphi_{\epsilon_m}(x - y)) = \varphi'_{\mathbb{R}}\left(\frac{|\tilde{x} - \tilde{y}|}{\epsilon_m}\right) \frac{y^i - x^i}{\epsilon_m |\tilde{x} - \tilde{y}|} \chi_{[1, n]}(i).$$

Hence,

$$\begin{aligned} |\partial_{y^i}(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y))| &\leq \left| \varphi'_{\mathbb{R}}\left(\frac{|\tilde{x} - \tilde{y}|}{\epsilon_m}\right) \frac{1}{\epsilon_m} - \varphi'_{\mathbb{R}}\left(\frac{|\tilde{x} - \tilde{y}|}{\epsilon_{m+1}}\right) \frac{1}{\epsilon_{m+1}} \right| \\ &\leq \left| \varphi'_{\mathbb{R}}\left(\frac{|\tilde{x} - \tilde{y}|}{\epsilon_m}\right) \right| \left| \frac{1}{\epsilon_m} - \frac{1}{\epsilon_{m+1}} \right| + \left| \varphi'_{\mathbb{R}}\left(\frac{|\tilde{x} - \tilde{y}|}{\epsilon_m}\right) - \varphi'_{\mathbb{R}}\left(\frac{|\tilde{x} - \tilde{y}|}{\epsilon_{m+1}}\right) \right| \frac{1}{\epsilon_{m+1}} \\ &\leq \left(\|\varphi'_{\mathbb{R}}\|_{\infty} + \|\varphi''_{\mathbb{R}}\|_{\infty} \frac{|\tilde{x} - \tilde{y}|}{\epsilon_{m+1}} \right) \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m \epsilon_{m+1}}. \end{aligned}$$

Since $\epsilon_m, \epsilon_{m+1} \in I_j$, we deduce from the previous estimate that, for $x - y \in \text{supp } \varphi_{\epsilon_{m+1}}^{\epsilon_m}$,

$$(5.6) \quad |\nabla_y(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y))| \lesssim \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m \epsilon_{m+1}} \approx 2^{2j} |\epsilon_m - \epsilon_{m+1}|.$$

We are going to use (5.5) and (5.6) to prove (5.3) and (5.4). Let us start with (5.3). Since $\epsilon_m, \epsilon_{m+1} \in I_j$, we can assume that $\text{supp } \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot) \subset B_D$, by taking C_Γ big enough.

By (5.5) and (5.6), for all $y \in \text{supp } \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot)$,

$$\left| \nabla_y(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y)K(x - y)) \right| \lesssim 2^{j(n+2)} |\epsilon_m - \epsilon_{m+1}|,$$

hence

$$|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (\mu - \sigma_D))(x)| \lesssim 2^{j(n+2)} |\epsilon_m - \epsilon_{m+1}| \text{dist}_{B_D}(\mu, \sigma_D) \lesssim 2^j |\epsilon_m - \epsilon_{m+1}| \alpha(D),$$

which gives (5.3).

In order to prove (5.4), set $L_D^x = \{(\tilde{t}, a(\tilde{t})) \in \mathbb{R}^d : \tilde{t} \in \mathbb{R}^n\}$, where $a : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$ is an appropriate affine map, and let $p : L_D \rightarrow L_D^x$ be the map defined by $p(t) := (\tilde{t}, a(\tilde{t}))$. Since Γ is a Lipschitz graph, a is well defined and p is a homeomorphism. Let $p_{\sharp} \mathcal{H}_{L_D}^n$ be the image measure of $\mathcal{H}_{L_D}^n$ by p . It is easy to see that, $|y - p(y)| \approx \text{dist}(y, L_D)$ for all $y \in L_D$. Notice also that the image measure $p_{\sharp} \mathcal{H}_{L_D}^n$ coincides with $\mathcal{H}_{L_D^x}^n$. Therefore, since $\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y)$ only

depends on $\tilde{x} - \tilde{y}$,

$$(5.7) \quad \begin{aligned} (K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (\sigma_D - \sigma_D^x))(x) &= c_D \int \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)K(x-y) d(\mathcal{H}_{L_D}^n - p_{\sharp}\mathcal{H}_{L_D}^n)(y) \\ &= c_D \int \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)(K(x-y) - K(x-p(y))) d\mathcal{H}_{L_D}^n(y). \end{aligned}$$

For $y \in \text{supp } \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot) \cap L_D$, we have

$$|K(x-y) - K(x-p(y))| \lesssim 2^{j(n+1)}|y-p(y)| \approx 2^{j(n+1)}\text{dist}(x, L_D).$$

Plugging this estimate and (5.5) into (5.7), we conclude that

$$|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (\sigma_D - \sigma_D^x))(x)| \lesssim 2^{2j}|\epsilon_m - \epsilon_{m+1}|\text{dist}(x, L_D),$$

which gives (5.4); and the theorem follows.

6. L^2 LOCALIZATION OF $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ AND $\mathcal{O} \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$

From here till the end of the paper, $\Gamma := \{x \in \mathbb{R}^d : x = (\tilde{x}, A(\tilde{x}))\}$ will be the graph of a Lipschitz function $A : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$, without any assumption on the support of A .

Theorem 6.1. *Let $\rho > 2$. There exist $C_1, C_2 > 0$ such that, for every $f \in L^\infty(\mathcal{H}_\Gamma^n)$ supported in $\Gamma \cap D$ (where $D := \tilde{D} \times \mathbb{R}^{d-n}$ and \tilde{D} is a cube of \mathbb{R}^n),*

$$(6.1) \quad \int_D ((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f)^2 d\mathcal{H}_\Gamma^n \leq C_1 \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)}^2 \mathcal{H}_\Gamma^n(D) \quad \text{and}$$

$$(6.2) \quad \int_D ((\mathcal{O} \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f)^2 d\mathcal{H}_\Gamma^n \leq C_2 \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)}^2 \mathcal{H}_\Gamma^n(D).$$

The constant C_2 does not depend on the fixed sequence that defines \mathcal{O} .

We will only give the proof of (6.1), because the proof of (6.2) follows by very similar arguments and computations.

We claim that it is enough to prove (6.1) for all functions f such that $f(x) \approx 1$ for all $x \in \Gamma \cap D$. Otherwise, we consider $g(x) := \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)}^{-1} f(x) + 2\chi_D(x)$, which clearly satisfies $g(x) \approx 1$ for all $x \in \Gamma \cap D$. Since $f = \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)}(g - 2\chi_D)$,

$$(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f(x) \leq \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)}((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})g(x) + 2(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})\chi_D(x)).$$

Applying (6.1) to the functions g and χ_D , we finally get

$$\int_D ((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f)^2 d\mathcal{H}_\Gamma^n \lesssim \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)}^2 \mathcal{H}_\Gamma^n(D).$$

Given f and D as in Theorem 6.1, from now on, we assume that $f \approx 1$ in $\Gamma \cap D$. Let \tilde{z}_D be the center of \tilde{D} and set $z_D := (\tilde{z}_D, A(\tilde{z}_D))$. One can easily construct a Lipschitz function $A_D : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$ such that $\text{Lip}(A_D) \lesssim \text{Lip}(A)$, $A_D(\tilde{x}) = A(\tilde{z}_D)$ for all $\tilde{x} \in (3\tilde{D})^c$, and $A_D(\tilde{x}) = A(\tilde{x})$ for all $\tilde{x} \in \tilde{D}$. Let Γ_D be the graph of A_D and define the measure $\mu := \mathcal{H}_{\Gamma_D \setminus D}^n + f\mathcal{H}_{\Gamma_D \cap D}^n$. Notice that $\chi_{(3D)^c}\mu$ is supported in the n -plane $L := \mathbb{R}^n \times \{A(\tilde{z}_D)\}$ and $\chi_D\mu = f\mathcal{H}_{\Gamma \cap D}^n$.

Since $f \approx 1$ in $\Gamma \cap D$ and $\chi_D\mu = (1 - \chi_{(3D)^c} - \chi_{3D \setminus D})\mu$, we can decompose

$$\begin{aligned} \int_D ((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f)^2 d\mathcal{H}_\Gamma^n &\approx \int_D \mathcal{V}_\rho(K\varphi * (\chi_D\mu))^2 d\mu \\ &\lesssim \int_D (\mathcal{V}_\rho(K\varphi * \mu) + \mathcal{V}_\rho(K\varphi * (\chi_{(3D)^c}\mu)) + \mathcal{V}_\rho(K\varphi * (\chi_{3D \setminus D}\mu)))^2 d\mu. \end{aligned}$$

In the next subsections, we will see that $\int_D \mathcal{V}_\rho(K\varphi * \mu)^2 d\mu$, $\int_D \mathcal{V}_\rho(K\varphi * (\chi_{(3D)^c} \mu))^2 d\mu$, and $\int_D \mathcal{V}_\rho(K\varphi * (\chi_{3D \setminus D} \mu))^2 d\mu$ are bounded by $C\mu(D)$, and (6.1) will follow.

6.1. Proof of $\int_D \mathcal{V}_\rho(K\varphi * \mu)^2 d\mu \lesssim \mu(D)$. Fix $x \in \text{supp} \mu$, and let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$(6.3) \quad (\mathcal{V}_\rho(K\varphi * \mu)(x))^\rho \leq 2 \sum_{m \in \mathbb{Z}} |(K\varphi_{\epsilon_{m+1}} * \mu)(x)|^\rho.$$

For $j \in \mathbb{Z}$ we set $I_j := [2^{-j-1}, 2^{-j})$. We decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$, where

$$(6.4) \quad \begin{aligned} \mathcal{S} &:= \bigcup_{j \in \mathbb{Z}} \mathcal{S}_j, \quad \mathcal{S}_j := \{m \in \mathbb{Z} : \epsilon_m, \epsilon_{m+1} \in I_j\}, \\ \mathcal{L} &:= \{m \in \mathbb{Z} : \epsilon_m \in I_i, \epsilon_{m+1} \in I_j \text{ for } i < j\}. \end{aligned}$$

Then, $\sum_{m \in \mathbb{Z}} |(K\varphi_{\epsilon_{m+1}} * \mu)(x)|^\rho = \sum_{m \in \mathcal{S}} |(K\varphi_{\epsilon_{m+1}} * \mu)(x)|^\rho + \sum_{m \in \mathcal{L}} |(K\varphi_{\epsilon_{m+1}} * \mu)(x)|^\rho$.

Notice that, since the $\ell^\rho(\mathbb{Z})$ -norm is smaller than the $\ell^2(\mathbb{Z})$ -norm for $\rho > 2$,

$$(6.5) \quad \sum_{m \in \mathcal{S}} |(K\varphi_{\epsilon_{m+1}} * \mu)(x)|^\rho \leq S\mu(x)^\rho,$$

where $S\mu(x)$ has been defined in Theorem 3.2.

Let us now estimate the sum over the indices $m \in \mathcal{L}$. For $m \in \mathbb{Z}$ we define $j(\epsilon_m)$ as the integer such that $\epsilon_m \in I_{j(\epsilon_m)}$. Since $\{\epsilon_m\}_{m \in \mathbb{Z}}$ is decreasing, given $j \in \mathbb{Z}$, there is at most one index $m \in \mathcal{L}$ such that $\epsilon_m \in I_j$. Thus, if $k, m \in \mathcal{L}$ and $k < m$, one has $j(\epsilon_k) < j(\epsilon_m)$.

With this notation, we have

$$(6.6) \quad \begin{aligned} \sum_{m \in \mathcal{L}} |(K\varphi_{\epsilon_{m+1}} * \mu)(x)|^\rho &= \sum_{m \in \mathcal{L}} |(K\varphi_{\epsilon_{m+1}} * \mu)(x) - (K\varphi_{\epsilon_m} * \mu)(x)|^\rho \\ &\lesssim \sum_{m \in \mathcal{L}} |(K\varphi_{\epsilon_{m+1}} * \mu)(x) - (K\varphi_{2^{-j(\epsilon_{m+1})-1}} * \mu)(x)|^\rho \\ &\quad + \sum_{m \in \mathcal{L}} |(K\varphi_{2^{-j(\epsilon_{m+1})-1}} * \mu)(x) - E_{j(\epsilon_{m+1})+1} \mu(x)|^\rho \\ &\quad + \sum_{m \in \mathcal{L}} |E_{j(\epsilon_{m+1})+1} \mu(x) - E_{j(\epsilon_m)+1} \mu(x)|^\rho \\ &\quad + \sum_{m \in \mathcal{L}} |E_{j(\epsilon_m)+1} \mu(x) - (K\varphi_{2^{-j(\epsilon_m)-1}} * \mu)(x)|^\rho \\ &\quad + \sum_{m \in \mathcal{L}} |(K\varphi_{2^{-j(\epsilon_m)-1}} * \mu)(x) - (K\varphi_{\epsilon_m} * \mu)(x)|^\rho \\ &\lesssim S\mu(x)^\rho + W\mu(x)^\rho + \mathcal{V}_\rho(E\mu)(x)^\rho, \end{aligned}$$

where $S\mu(x)$ and $W\mu(x)$ have been defined in Theorems 3.2 and 3.1, respectively, and $\mathcal{V}_\rho(E\mu)$ is the ρ -variation of the average of martingales $\{E_m \mu\}_{m \in \mathbb{Z}}$ from subsection 2.3. Therefore, by (6.5), (6.6), and (6.3), we deduce that

$$\mathcal{V}_\rho(K\varphi * \mu)(x) \lesssim S\mu(x) + W\mu(x) + \mathcal{V}_\rho(E\mu)(x)$$

for all $x \in \text{supp} \mu$, and so

$$(6.7) \quad \int_D \mathcal{V}_\rho(K\varphi * \mu)^2 d\mu \lesssim \|S\mu\|_{L^2(\mu)}^2 + \|W\mu\|_{L^2(\mu)}^2 + \|\mathcal{V}_\rho(E\mu)\|_{L^2(\mu)}^2.$$

Clearly, Theorem 2.7, Theorem 3.1, and Theorem 3.2 can be applied to the measure μ , because $\text{supp}\mu$ is a translation of the graph of a Lipschitz function with compact support. These theorems in combination with (6.7) yield

$$(6.8) \quad \int_D \mathcal{V}_\rho(K\varphi * \mu)^2 d\mu \leq C_1 \left(\mu(3D) + \sum_{Q \in \mathcal{D}} (\alpha_\mu(C_2 Q)^2 + \beta_{2,\mu}(Q)^2) \mu(Q) \right),$$

where $C_1, C_2 > 0$ only depend on $n, d, K, \text{Lip}(A)$, and ρ (the condition $\rho > 2$ is used to ensure the L^2 boundedness of $\mathcal{V}_\rho(E\mu)$). Obviously, $\mu(3D) \approx \mu(D)$ and, since $\chi_{(3D)^c}\mu$ coincides with the n -dimensional Hausdorff measure on an n -plane, using Remark 2.2 it is easy to check that $\sum_{Q \in \mathcal{D}} (\alpha_\mu(C_2 Q)^2 + \beta_{2,\mu}(Q)^2) \mu(Q) \lesssim \mu(3D)$. Hence, we conclude that $\int_D \mathcal{V}_\rho(K\varphi * \mu)^2 d\mu \lesssim \mu(D)$ by (6.8).

6.2. Proof of $\int_D \mathcal{V}_\rho(K\varphi * (\chi_{(3D)^c}\mu))^2 d\mu \lesssim \mu(D)$. Fix $x \in \text{supp}\mu \cap D$, and let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$(6.9) \quad (\mathcal{V}_\rho(K\varphi * (\chi_{(3D)^c}\mu))(x))^\rho \leq 2 \sum_{m \in \mathbb{Z}} |(K\varphi_{\epsilon_{m+1}}^\epsilon * (\chi_{(3D)^c}\mu))(x)|^\rho.$$

Recall that \tilde{z}_D is the center of \tilde{D} , $z_D := (\tilde{z}_D, A(\tilde{z}_D))$ and $L := \mathbb{R}^n \times \{A(\tilde{z}_D)\}$. Since $\chi_{(3D)^c}\mu = \mathcal{H}_{L \cap 3D}^n$ and z_D is the center of $L \cap D$, $(K\varphi_\epsilon^\delta * (\chi_{(3D)^c}\mu))(z_D) = 0$ for all $0 < \epsilon \leq \delta$. Thus, $|(K\varphi_{\epsilon_{m+1}}^\epsilon * (\chi_{(3D)^c}\mu))(x)| = |(K\varphi_{\epsilon_{m+1}}^\epsilon * (\chi_{(3D)^c}\mu))(x) - (K\varphi_{\epsilon_{m+1}}^\epsilon * (\chi_{(3D)^c}\mu))(z_D)| \leq \Theta 1_m + \Theta 2_m$, where

$$(6.10) \quad \begin{aligned} \Theta 1_m &:= \int_{(3D)^c} \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) |K(x-y) - K(z_D-y)| d\mu(y), \\ \Theta 2_m &:= \int_{(3D)^c} |\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) - \varphi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)| K(z_D-y) d\mu(y). \end{aligned}$$

Since $x \in \text{supp}\mu \cap D$ and A is a Lipschitz function, we have $|x - z_D| \lesssim \ell(D)$, and then $|K(x-y) - K(z_D-y)| \lesssim |x - z_D| |z_D - y|^{-n-1} \lesssim \ell(D) |z_D - y|^{-n-1}$ for all $y \in (3D)^c$. Therefore, using that $\sum_{m \in \mathbb{Z}} \varphi_{\epsilon_{m+1}}^{\epsilon_m} \leq 1$ and that $\rho > 1$,

$$(6.11) \quad \left(\sum_{m \in \mathbb{Z}} \Theta 1_m^\rho \right)^{1/\rho} \leq \sum_{m \in \mathbb{Z}} \Theta 1_m \lesssim \int_{(3D)^c} \ell(D) |z_D - y|^{-n-1} d\mu(y) \lesssim 1.$$

To deal with $\Theta 2_m$, we decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$ as in (6.4). As before, given $m \in \mathbb{Z}$, let $j(\epsilon_m)$ be the integer such that $\epsilon_m \in I_{j(\epsilon_m)}$. Observe that

$$\text{supp} \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot) \subset A(\tilde{x}, 2.1\sqrt{n}2^{-j(\epsilon_{m+1})-1}, 3\sqrt{n}2^{-j(\epsilon_m)}) \times \mathbb{R}^{d-n} =: A_m(x).$$

Notice also that the sets $A_m(x)$ have finite overlap for $m \in \mathcal{L}$, and the same is true for the sets $A'_j(x) := A(\tilde{x}, 2.1\sqrt{n}2^{-j-1}, 3\sqrt{n}2^{-j}) \times \mathbb{R}^{d-n}$ for $j \in \mathbb{Z}$. The same observations hold if we replace x by z_D (and \tilde{x} by \tilde{z}_D). Obviously, $A_m(x) \subset A'_j(x)$ (and $A_m(z_D) \subset A'_j(z_D)$) for $m \in \mathcal{S}_j$.

Assume that $m \in \mathcal{S}$. With the same computations as those carried out in (5.6), one can easily prove that, for all $z - y \in \text{supp} \varphi_{\epsilon_{m+1}}^{\epsilon_m}$,

$$|\nabla_z(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(z-y))| \lesssim \left(\|\varphi'_{\mathbb{R}}\|_{L^\infty(\mathbb{R})} + \|\varphi''_{\mathbb{R}}\|_{L^\infty(\mathbb{R})} \frac{|\tilde{z} - \tilde{y}|}{\epsilon_{m+1}} \right) \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_m \epsilon_{m+1}} \lesssim 2^{j(\epsilon_m)} \frac{|\epsilon_m - \epsilon_{m+1}|}{|\tilde{z} - \tilde{y}|},$$

because $|\tilde{z} - \tilde{y}| \approx \epsilon_m \approx \epsilon_{m+1} \approx 2^{j(\epsilon_m)}$ for all $z - y \in \text{supp} \varphi_{\epsilon_{m+1}}^{\epsilon_m}$ and $m \in \mathcal{S}$. In particular, if $z \in D$ and $y \in (3D)^c$, $|\nabla_z(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(z - y))| \lesssim 2^{j(\epsilon_m)} |\epsilon_m - \epsilon_{m+1}| |\tilde{z}_D - \tilde{y}|^{-1}$. Hence,

$$\Theta 2_m \lesssim \int_{(A_m(x) \cup A_m(z_D)) \setminus 3D} \ell(D) 2^{j(\epsilon_m)} \frac{|\epsilon_m - \epsilon_{m+1}|}{|\tilde{z}_D - \tilde{y}|^{n+1}} d\mu(y),$$

and then,

$$\begin{aligned} \left(\sum_{m \in \mathcal{S}} \Theta 2_m^\rho \right)^{1/\rho} &\lesssim \sum_{m \in \mathcal{S}} \int_{(A_m(x) \cup A_m(z_D)) \setminus 3D} \ell(D) 2^{j(\epsilon_m)} \frac{|\epsilon_m - \epsilon_{m+1}|}{|\tilde{z}_D - \tilde{y}|^{n+1}} d\mu(y) \\ (6.12) \quad &\leq \sum_{j \in \mathbb{Z}} \int_{(A'_j(x) \cup A'_j(z_D)) \setminus 3D} \frac{\ell(D)}{|\tilde{z}_D - \tilde{y}|^{n+1}} \sum_{m \in \mathcal{S}_j} \frac{|\epsilon_m - \epsilon_{m+1}|}{2^{-j}} d\mu(y) \\ &\lesssim \int_{(3D)^c} \frac{\ell(D)}{|\tilde{z}_D - \tilde{y}|^{n+1}} d\mu(y) \lesssim 1. \end{aligned}$$

Assume now that $m \in \mathcal{L}$. It is easy to check that $|\nabla_z(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(z - y))| \lesssim |\tilde{z} - \tilde{y}|^{-1}$ for all $z, y \in \mathbb{R}^d$. So, if also $z \in D$ and $y \in (3D)^c$, $|\nabla_z(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(z - y))| \lesssim |\tilde{z}_D - \tilde{y}|^{-1}$. Therefore,

$$\begin{aligned} \left(\sum_{m \in \mathcal{L}} \Theta 2_m^\rho \right)^{1/\rho} &\lesssim \sum_{m \in \mathcal{L}} \int_{(A_m(x) \cup A_m(z_D)) \setminus 3D} \frac{\ell(D)}{|z_D - y|^{n+1}} d\mu(y) \\ (6.13) \quad &\lesssim \int_{(3D)^c} \frac{\ell(D)}{|\tilde{z}_D - \tilde{y}|^{n+1}} d\mu(y) \lesssim 1. \end{aligned}$$

Finally combining (6.11), (6.12), and (6.13), with (6.9) and the fact that $(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (\chi_{(3D)^c}\mu))(x) \leq \Theta 1_m + \Theta 2_m$, we conclude that

$$\mathcal{V}_\rho(K\varphi * (\chi_{(3D)^c}\mu))(x) \lesssim \left(\sum_{m \in \mathbb{Z}} \Theta 1_m^\rho \right)^{1/\rho} + \left(\sum_{m \in \mathcal{S}} \Theta 2_m^\rho \right)^{1/\rho} + \left(\sum_{m \in \mathcal{L}} \Theta 2_m^\rho \right)^{1/\rho} \lesssim 1$$

for all $x \in \text{supp} \mu \cap D$. Therefore, $\int_D \mathcal{V}_\rho(K\varphi * (\chi_{(3D)^c}\mu))^2 d\mu \lesssim \mu(D)$.

6.3. Proof of $\int_D \mathcal{V}_\rho(K\varphi * (\chi_{3D \setminus D}\mu))^2 d\mu \lesssim \mu(D)$. Fix $x \in \text{supp} \mu \cap D$. Since $\rho > 1$,

$$\begin{aligned} \mathcal{V}_\rho(K\varphi * (\chi_{3D \setminus D}\mu))(x) &= \sup_{\{\epsilon_m\}} \left(\sum_{m \in \mathbb{Z}} \left| \int_{3D \setminus D} \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y) K(x - y) d\mu(y) \right|^\rho \right)^{1/\rho} \\ &\leq \sup_{\{\epsilon_m\}} \sum_{m \in \mathbb{Z}} \int_{3D \setminus D} \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y) |K(x - y)| d\mu(y) \leq \int_{3D \setminus D} |K(x - y)| d\mu(y). \end{aligned}$$

By a standard computation, one can show that

$$\int_D \left(\int_{3D \setminus D} |K(x - y)| d\mu(y) \right)^2 d\mu(x) \lesssim \mu(D),$$

hence we conclude that $\int_D \mathcal{V}_\rho(K\varphi * (\chi_{3D \setminus D}\mu))^2 d\mu \lesssim \mu(D)$.

7. L^p AND ENDPOINT ESTIMATES FOR $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ AND $\mathcal{O} \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$

We denote by $H^1(\mathcal{H}_\Gamma^n)$ and $BMO(\mathcal{H}_\Gamma^n)$ the (atomic) Hardy space and the space of functions with bounded mean oscillation, respectively, with respect to the measure \mathcal{H}_Γ^n . These spaces are defined as the classical $H^1(\mathbb{R}^d)$ and $BMO(\mathbb{R}^d)$ (see [Du, Chapter 6], for example), but by replacing the true cubes of \mathbb{R}^d by our special v-cubes.

Theorem 7.1. *Let $\rho > 2$. The operators $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ and $\mathcal{O} \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ are bounded*

- in $L^p(\mathcal{H}_\Gamma^n)$ for $1 < p < \infty$,
- from $H^1(\mathcal{H}_\Gamma^n)$ to $L^1(\mathcal{H}_\Gamma^n)$,
- from $L^1(\mathcal{H}_\Gamma^n)$ to $L^{1,\infty}(\mathcal{H}_\Gamma^n)$, and
- from $L^\infty(\mathcal{H}_\Gamma^n)$ to $BMO(\mathcal{H}_\Gamma^n)$,

and the norm of $\mathcal{O} \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ in the cases above is bounded independently of the sequence that defines \mathcal{O} .

We will only give the proof of Theorem 7.1 in the case of the ρ -variation, because the proof for the oscillation follows by analogous arguments.

7.1. The operator $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n} : H^1(\mathcal{H}_\Gamma^n) \rightarrow L^1(\mathcal{H}_\Gamma^n)$ is bounded. Fix a cube $\tilde{D} \subset \mathbb{R}^n$ and set $D := \tilde{D} \times \mathbb{R}^{d-n}$. Let f be an atom, i.e., a function defined on Γ and such that

$$(7.1) \quad \text{supp } f \subset D, \quad \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} \leq \frac{1}{\mathcal{H}_\Gamma^n(D)}, \quad \text{and} \quad \int f d\mathcal{H}_\Gamma^n = 0.$$

We have to prove that $\int (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f d\mathcal{H}_\Gamma^n \leq C$, for some constant $C > 0$ which does not depend on f or D . Since $(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f(x)$ is well defined and non negative for $f \in L^1(\mathcal{H}_\Gamma^n)$, the uniform boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ on atoms yields its boundedness from $H^1(\mathcal{H}_\Gamma^n)$ to $L^1(\mathcal{H}_\Gamma^n)$ by standard arguments. We omit the details.

First of all, by Hölder's inequality, Theorem 6.1, and (7.1),

$$\begin{aligned} \int_{3D} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f d\mathcal{H}_\Gamma^n &\leq \mathcal{H}_\Gamma^n(3D)^{1/2} \left(\int_{3D} ((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f)^2 d\mathcal{H}_\Gamma^n \right)^{1/2} \\ &\lesssim \mathcal{H}_\Gamma^n(3D)^{1/2} \left(\|f\|_{L^\infty(\mathcal{H}_\Gamma^n)}^2 \mathcal{H}_\Gamma^n(3D) \right)^{1/2} \lesssim 1. \end{aligned}$$

Thus, it remains to prove that $\int_{(3D)^c} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f d\mathcal{H}_\Gamma^n \leq C$.

Given $x \in \Gamma \setminus 3D$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$(7.2) \quad ((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f(x))^\rho \leq 2 \sum_{m \in J} |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mathcal{H}_\Gamma^n))(x)|^\rho,$$

where $J := \{m \in \mathbb{Z} : \text{supp } \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot) \cap \text{supp } f \neq \emptyset\}$.

Set $z_D := (\tilde{z}_D, A(\tilde{z}_D)) \in D \cap \Gamma$, where \tilde{z}_D is the center of \tilde{D} . By (7.1), we have $\int \varphi_\epsilon^\delta(x - z_D) K(x - z_D) f(y) d\mathcal{H}_\Gamma^n(y) = 0$ for all $0 < \epsilon \leq \delta$. Thus, given $m \in J$, we can decompose

$$\begin{aligned} (K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mathcal{H}_\Gamma^n))(x) &= \int \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y) (K(x - y) - K(x - z_D)) f(y) d\mathcal{H}_\Gamma^n(y) \\ &\quad + \int (\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y) - \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - z_D)) K(x - z_D) f(y) d\mathcal{H}_\Gamma^n(y), \end{aligned}$$

and we obtain $|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f\mathcal{H}_\Gamma^n))(x)| \leq \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} (\Theta 1_m + \Theta 2_m)$, where

$$\begin{aligned} \Theta 1_m &:= \int_D \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y) |K(x - y) - K(x - z_D)| d\mathcal{H}_\Gamma^n(y), \\ \Theta 2_m &:= \int_D |\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y) - \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - z_D)| |K(x - z_D)| d\mathcal{H}_\Gamma^n(y). \end{aligned}$$

The term $\Theta 1_m$ can be easily handled. For $x \in \Gamma \setminus 3D$, we have

$$(7.3) \quad \Theta 1_m \lesssim \ell(D) \operatorname{dist}(x, D)^{-n-1} \int_D \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) d\mathcal{H}_\Gamma^n(y).$$

Let us estimate $\Theta 2_m$. Decompose $J = \mathcal{S} \cup \mathcal{L}$, where \mathcal{S} and \mathcal{L} are as in (6.4) but replacing $m \in \mathbb{Z}$ by $m \in J$, and as before, let $j(\epsilon_m)$ be the integer such that $\epsilon_m \in I_{j(\epsilon_m)}$. Using that $x \in \Gamma \setminus 3D$ and $\operatorname{supp} f \subset D$, one can easily check that \mathcal{L} contains a finite number of elements, and this number only depends on n and d . Similarly, $\mathcal{S}_j = \emptyset$ for all $j \in \mathbb{Z}$ except on a finite number which only depends on n and d .

Assume that $m \in \mathcal{S}$. With the same computations as those in (5.6), one can prove that, for all $y \in \operatorname{supp} \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot)$, $|\nabla_y(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y))| \lesssim 2^{j(\epsilon_m)} |\epsilon_m - \epsilon_{m+1}| |\tilde{x} - \tilde{y}|^{-1}$, because $|\tilde{x} - \tilde{y}| \approx \epsilon_m \approx \epsilon_{m+1} \approx 2^{-j(\epsilon_m)}$ for all $y \in \operatorname{supp} \varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - \cdot)$. Thus,

$$(7.4) \quad \Theta 2_m \lesssim \ell(D)^{n+1} \operatorname{dist}(x, D)^{-n-1} 2^{j(\epsilon_m)} |\epsilon_m - \epsilon_{m+1}|.$$

Assume now that $m \in \mathcal{L}$. It is easy to verify that $|\nabla_y(\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x - y))| \lesssim |\tilde{x} - \tilde{y}|^{-1}$, so $\Theta 2_m \lesssim \ell(D)^{n+1} \operatorname{dist}(x, D)^{-n-1}$.

Combining this last estimate with (7.3), (7.4), the fact that $|(K \varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f \mathcal{H}_\Gamma^n))(x)| \leq \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} (\Theta 1_m + \Theta 2_m)$, the remark on \mathcal{S} and \mathcal{L} made just after (7.3), (7.2), and that $\rho > 1$, we have that, for all $x \in \Gamma \setminus 3D$,

$$\begin{aligned} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f(x) &\lesssim \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} \left(\sum_{m \in J} \Theta 1_m + \sum_{m \in \mathcal{S}} \Theta 2_m + \sum_{m \in \mathcal{L}} \Theta 2_m \right) \\ &\lesssim \frac{\|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} \ell(D)^{n+1}}{\operatorname{dist}(x, D)^{n+1}} \left(\sum_{m \in J} \int_D \frac{\varphi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)}{\ell(D)^n} d\mathcal{H}_\Gamma^n(y) + \sum_{m \in \mathcal{S}} \frac{|\epsilon_m - \epsilon_{m+1}|}{2^{-j(\epsilon_m)}} + \sum_{m \in \mathcal{L}} 1 \right) \\ &\lesssim \frac{\|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} \ell(D)^{n+1}}{\operatorname{dist}(x, D)^{n+1}}. \end{aligned}$$

Then, using (7.1) and standard estimates, we conclude that

$$\int_{(3D)^c} (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f(x) d\mathcal{H}_\Gamma^n(x) \lesssim \int_{(3D)^c} \frac{\|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} \ell(D)^{n+1}}{\operatorname{dist}(x, D)^{n+1}} d\mathcal{H}_\Gamma^n(x) \lesssim 1.$$

7.2. The operator $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n} : L^\infty(\mathcal{H}_\Gamma^n) \rightarrow BMO(\mathcal{H}_\Gamma^n)$ is bounded. We have to prove that there exists a constant $C > 0$ such that, for any $f \in L^\infty(\mathcal{H}_\Gamma^n)$ and any cube $\tilde{D} \subset \mathbb{R}^n$, there exists some constant c depending on f and \tilde{D} such that

$$\int_D |(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f - c| d\mathcal{H}_\Gamma^n \leq C \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} \mathcal{H}_\Gamma^n(D).$$

Let f and D be as above, and set $f_1 := f \chi_{3D}$ and $f_2 := f - f_1$. First of all, by Hölder's inequality and Theorem 6.1, we have

$$\begin{aligned} \int_D (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f_1 d\mathcal{H}_\Gamma^n &\leq \mathcal{H}_\Gamma^n(D)^{1/2} \left(\int_{3D} ((\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}) f_1)^2 d\mathcal{H}_\Gamma^n \right)^{1/2} \\ &\lesssim \mathcal{H}_\Gamma^n(D)^{1/2} \left(\|f_1\|_{L^\infty(\mathcal{H}_\Gamma^n)}^2 \mathcal{H}_\Gamma^n(3D) \right)^{1/2} \lesssim \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} \mathcal{H}_\Gamma^n(D). \end{aligned}$$

Notice that $|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})(f_1 + f_2) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_2| \leq (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_1$, because $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ is sublinear and positive. Then, for any $c \in \mathbb{R}$,

$$\begin{aligned} |(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})(f_1 + f_2) - c| &\leq |(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})(f_1 + f_2) - (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_2| + |(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_2 - c| \\ &\leq (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_1 + |(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_2 - c|, \end{aligned}$$

hence we are reduced to prove that, for some constant $c \in \mathbb{R}$,

$$(7.5) \quad \int_D |(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_2 - c| d\mathcal{H}_\Gamma^n \leq C \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)} \mathcal{H}_\Gamma^n(D).$$

Set $z_D := (\tilde{z}_D, A(\tilde{z}_D))$, where z_D is the center of \tilde{D} , and take $c := (\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_2(z_D)$. We may assume that $c < \infty$ (this is the case if, for example, f has compact support).

Given a family $a := \{a_\epsilon\}_{\epsilon>0} \subset \mathbb{C}$, define its ρ -variation semi-norm to be

$$\|a\|_{\mathcal{V}_\rho} = \sup_{\{\epsilon_m\}} \left(\sum_{m \in \mathbb{Z}} |a_{\epsilon_{m+1}} - a_{\epsilon_m}|^\rho \right)^{1/\rho},$$

where the supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m \in \mathbb{Z}} \subset (0, \infty)$. Since $\|\cdot\|_{\mathcal{V}_\rho}$ satisfies the triangle inequality, we have $|\|a\|_{\mathcal{V}_\rho} - \|b\|_{\mathcal{V}_\rho}| \leq \|a - b\|_{\mathcal{V}_\rho}$ for all $a := \{a_\epsilon\}_{\epsilon>0}$ and $b := \{b_\epsilon\}_{\epsilon>0}$, where $a - b := \{a_\epsilon - b_\epsilon\}_{\epsilon>0}$. Hence,

$$|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_2(x) - c|^\rho \leq \sup_{\{\epsilon_m \searrow 0\}} \sum_{m \in \mathbb{Z}} |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f_2 \mathcal{H}_\Gamma^n))(x) - (K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f_2 \mathcal{H}_\Gamma^n))(z_D)|^\rho.$$

Given $x \in \Gamma \cap D$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_2(x) - c|^\rho \leq 2 \sum_{m \in \mathbb{Z}} |(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f_2 \mathcal{H}_\Gamma^n))(x) - (K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f_2 \mathcal{H}_\Gamma^n))(z_D)|^\rho.$$

Notice that $|(K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f_2 \mathcal{H}_\Gamma^n))(x) - (K\varphi_{\epsilon_{m+1}}^{\epsilon_m} * (f_2 \mathcal{H}_\Gamma^n))(z_D)| \leq \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)}(\Theta 1_m + \Theta 2_m)$, where $\Theta 1_m$ and $\Theta 2_m$ are as in (6.10) but replacing μ by \mathcal{H}_Γ^n . It is straightforward to check that the arguments and computations given in subsection 6.2 to estimate the two terms in (6.10) (see (6.11), (6.12), and (6.13)) still hold if we replace μ by \mathcal{H}_Γ^n . Therefore, we have

$$\sum_{m \in \mathbb{Z} = S \cup \mathcal{L}} (\Theta 1_m + \Theta 2_m)^\rho \lesssim 1,$$

which implies that $|(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f_2(x) - c| \lesssim \|f\|_{L^\infty(\mathcal{H}_\Gamma^n)}$ and, by integrating in D , gives (7.5).

7.3. The operator $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n} : L^p(\mathcal{H}_\Gamma^n) \rightarrow L^p(\mathcal{H}_\Gamma^n)$ is bounded for all $1 < p < \infty$. Since $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ is sublinear, the L^p boundedness follows by applying the results of subsection 7.1 and subsection 7.2, and the interpolation theorem between the pairs $(H^1(\mathcal{H}_\Gamma^n), L^1(\mathcal{H}_\Gamma^n))$ and $(L^\infty(\mathcal{H}_\Gamma^n), BMO(\mathcal{H}_\Gamma^n))$ in [Ju, page 43].

Given a v-cube $Q \subset \mathbb{R}^d$, set $m_Q(f) := \mathcal{H}_\Gamma^n(Q)^{-1} \int_Q f d\mathcal{H}_\Gamma^n$, and let M denote the Hardy-Littlewood maximal operator with respect to Γ , i.e. for $x \in \Gamma$, $M(f)(x) := \sup m_Q(|f|)$, where the supremum is taken over all v-cubes $Q \subset \mathbb{R}^d$ containing $x \in \Gamma$. Let M^\sharp be the sharp maximal operator defined by $M^\sharp(f)(x) := \sup m_Q(|f - m_Q(f)|)$, where the supremum is also taken over all v-cubes $Q \subset \mathbb{R}^d$ containing $x \in \Gamma$.

One comment about the interpolation theorem in [Ju, page 43] is in order. Given an operator F bounded from H^1 to L^1 and from L^∞ to BMO , in the proof of the interpolation theorem applied to F , one uses that $M^\sharp \circ F$ is sublinear (i.e. $(M^\sharp \circ F)(f + g) \leq (M^\sharp \circ F)f +$

$(M^\sharp \circ F)g$ for all functions f, g . This is the case when F is linear. However, $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ is not linear, and then it is not clear if $M^\sharp \circ \mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ is sublinear. Nevertheless, this problem can be fixed easily using that $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ is sublinear and positive (that is $(\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n})f(x) \geq 0$ for all f and x), as the following lemma shows.

Lemma 7.2. *Let $F : L^1_{loc}(\mathcal{H}_\Gamma^n) \rightarrow L^1_{loc}(\mathcal{H}_\Gamma^n)$ be a positive and sublinear operator. Then $(M^\sharp \circ F)(f + g) \lesssim (M \circ F)f + (M^\sharp \circ F)g$ for all functions f, g .*

Proof. If F is sublinear and positive, one has that $|F(f)(x) - F(g)(x)| \leq F(f - g)(x)$ for all functions $f, g \in L^1_{loc}(\mathcal{H}_\Gamma^n)$. In particular, $|F(f + g)(x) - F(g)(x)| \leq F(f)(x)$. Then, for $x, y \in Q \cap \Gamma$,

$$\begin{aligned} |F(f + g)(y) - m_Q(Fg)| &\leq |F(f + g)(y) - Fg(y)| + |Fg(y) - m_Q(Fg)| \\ &\leq |Ff(y)| + |Fg(y) - m_Q(Fg)|. \end{aligned}$$

Hence, $m_Q|F(f + g) - m_Q(Fg)| \leq m_Q|Ff| + m_Q|Fg - m_Q(Fg)| \leq (M \circ F)f(x) + (M^\sharp \circ F)g(x)$ and, by taking the supremum over all possible v-cubes $Q \ni x$, we conclude $(M^\sharp \circ F)(f + g)(x) \lesssim (M \circ F)f(x) + (M^\sharp \circ F)g(x)$. \square

By using Lemma 7.2 and the fact that $\|Mf\|_{L^p(\mathcal{H}_\Gamma^n)} \lesssim \|M^\sharp f\|_{L^p(\mathcal{H}_\Gamma^n)}$ for $f \in L^{p_0}(\mathcal{H}_\Gamma^n) \cap L^p(\mathcal{H}_\Gamma^n)$ and $1 \leq p_0 \leq p < \infty$, one can reprove Journé's interpolation theorem applied to $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ with minor modifications in the original proof.

7.4. The operator $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n} : L^1(\mathcal{H}_\Gamma^n) \rightarrow L^{1,\infty}(\mathcal{H}_\Gamma^n)$ is bounded. By adapting [CJRW2, Theorem B] to our setting and using the smoothness of the family φ , one can show that the $L^2(\mathcal{H}_\Gamma^n)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{T}_\varphi^{\mathcal{H}_\Gamma^n}$ yields the boundedness of this operator from $L^1(\mathcal{H}_\Gamma^n)$ to $L^{1,\infty}(\mathcal{H}_\Gamma^n)$. The interested reader may see [MT], where a more general result is proved.

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